

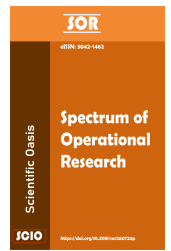


**SCIENTIFIC OASIS**

Spectrum of Operational Research

Journal homepage: [www.sor-journal.org](http://www.sor-journal.org)

ISSN: 3042-1470



# An Extension of VIKOR Approach for MCDM Using Bipolar Fuzzy Preference $\delta$ -Covering Based Bipolar Fuzzy Rough Set Model

Rizwan Gul<sup>1,\*</sup>

<sup>1</sup> Department of Mathematics, Quaid-i-Azam University, Islamabad, 45320, Pakistan

## ARTICLE INFO

### Article history:

Received 25 June 2024

Received in revised form 28 September 2024

Accepted 20 October 2024

Available online 27 October 2024

### Keywords:

Rough set; Bipolar fuzzy preference relation; BFP $\delta$ C-BFRSs; Decision-making.

## ABSTRACT

Bipolarity highlights the positive and negative facets of a certain dilemma. This script aims to propose a novel multi-criteria decision-making (MCDM) approach based on the bipolar fuzzy preference  $\delta$ -covering based bipolar fuzzy rough set (BFP $\delta$ C-BFRS) model by combining the VIKOR (ViseKriterijumska Optimizacija I Kompromisno Rasenje) scheme. The VIKOR scheme is viewed as a beneficial MCDM strategy, particularly in situations where an expert is incapable of making the right decision at the beginning of system design. The VIKOR method works well for problems with competing attributes because it operates under the presumptions that compromise is acceptable in conflict resolution, the expert seeks a solution that is extremely close to the best, and all developed attributes are taken into consideration when processing the various alternatives. In this study, firstly, we propose an integrated MCDM based on BFP $\delta$ C-BFRSs using the VIKOR methodology. Moreover, we solve a real-world illustration to show the effectiveness of the expanded VIKOR approach. Finally, we demonstrate a detailed comparative analysis of the proposed methodology with some prevalent decision-making approaches to substantiate the accountability of the recommended scheme.

## 1. Introduction

The actual world is full of indeterminacy and uncertainty. Rather than dealing with specific problems, we typically deal with ambiguous ones. The success of traditional techniques is not always guaranteed because of the inherent uncertainties in these issues. Zadeh [1] promulgated the paradigm of the fuzzy set (FS), which unlocked the avenues for scholars to combat the uncertainty of data. FS theory relies on membership function (MF), which permits us to assess the membership grade (MG) of an item regarding a set. The bigger the MG, the higher the association of that item to the related set. Many remarkable applications of FSs can be seen in [2-5].

\*Corresponding author.

E-mail address: [rgul@math.qau.edu.pk](mailto:rgul@math.qau.edu.pk)

<https://doi.org/10.31181/sor21202511>

© The Author(s) 2025 | [Creative Commons Attribution 4.0 International License](https://creativecommons.org/licenses/by/4.0/)

It has been acknowledged that RS theory [6, 7] is a useful mathematical approach to managing intelligent systems that display ambiguity and uncertainty. The effectiveness of this comparatively emerging soft computing tool has been effectively demonstrated and leveraged in a variety of fields, including pattern recognition, conflict resolution, knowledge discovery, data mining, image processing, medical science, neural networks, and so forth. It has garnered a lot of attention in recent years. The indiscernibility relation between arbitrary items is described by equivalence relation (ER), the central and fundamental idea of the RS theory. Although RS theory has been effectively utilized in many areas, certain issues might lead the theory's application scope to be limited. These flaws could be the consequence of inaccurate knowledge of the items being evaluated. It might be difficult to find an ER with a piece of incomplete information. Consequently, RS models have developed several fascinating generalizations under various circumstances in recent years, such as an RS based on tolerance relations [8], RS based on neighbourhood operators [9], fuzzy rough set (FRS) [10], rough fuzzy set (RFS) model [11], dominance-based RSs [12], variable precision RS [13], covering-based RSs [14] and grey tolerance RS [15].

It is critical to keep in mind that everything has two aspects and that fuzziness and bipolarity are innate characteristics of human perception. Cognitive psychology research indicates that bipolar reasoning is essential to human cognition actions. The areas of the brain responsible for positive and negative impacts do not appear to be situated there. Experts in a variety of fields have noted the importance of bipolarity, including database querying, decision-making, and categorization. Fuzziness and bipolarity are two separate but correlative concepts intended to mimic certain facets of human cognition. The latter emphasizes the polarity and importance of the facts, whereas the former concentrates on linguistic imprecision. Recent research has consistently demonstrated the great importance of two concepts. In light of this, Zhang [16] postulated bipolar FSs (BFSs) as an extension of FSs with an MG lies in  $[-1, 1]$ . An object's MG of "0" indicates that it does not pertain to the associated property; an object's MG of  $(0, 1]$  means that it partially fulfils the property; and an object's MG of  $[-1, 0)$  signifies that it partially satisfies the implicit counter-property. BFS theory has been widely applied to solve practical issues. Additionally, several initiatives have been undertaken to integrate RS and BFS theories. Han *et al.* [17] devised a bipolar-valued RFS version with application in decision analysis. Yang *et al.* [18] recommended a bipolar FRS model on dual universes. An MCDM technique employing interval-valued bipolar fuzzy information was first introduced by Wei *et al.* [19]. Ali *et al.* [20] studied attribute reductions of bipolar fuzzy relation (BFR). By using the  $(\alpha, \beta)$ -indiscernibility of a BFR, Gul and Shabir [21] proposed a unique method for the roughness of a crisp set. A bipolar FRS model with inconsistent bipolarity in two universes was proposed by Han *et al.* [22]. Jana and Pal [23] studied a robust bipolar fuzzy EDAS scheme. Malik and Shabir [24] launched a consensus approach via rough BFSs. Luo and Hu [25] integrated a bipolar three-way decision scheme with applicability in analyzing incomplete information. Recently, Gul *et al.* [26] devised a bipolar fuzzy preference  $\delta$ -covering based bipolar FRS (BFP $\delta$ C-BFRS) model and corresponding decision-making applications.

In light of the literature mentioned above, we observed that a plethora of researchers put forward several hybrid models of RSs, FS, and BFSs. However, to the best of our knowledge, no prior investigation has been conducted into the hybridization of VIKOR methodology and BFP $\delta$ C-BFRS model. To bridge this knowledge gap, in this study, we studied an extension of the VIKOR scheme for MCDM under the BFP $\delta$ C-BFRS model.

The rest of this study is systematized as follows: In Section 2, we revisit some rudimentary ideas linked to FSs, BFSs, and RSs. In Section 3, we first proposed an integrated MCDM scheme based on the BFP $\delta$ C-BFRS model and the VIKOR approach. Besides, an algorithm for MCDM is also elaborated. In Section 4, an applied example of decision-making using bipolar fuzzy information is provided. In Section 5, we perform a comparative study of our projected methodology with other prevailing schemes. Section 6 provides conclusions.

## 2. Preliminaries

In this segment, we review numerous essential terminologies which will be utilized in our study.

**Definition 2.1** [6] An ordered pair  $(\Upsilon, \sigma)$  is named as an approximation space, where  $\Upsilon$  is a non-void finite set and  $\sigma$  is an ER over  $\Upsilon$ . For  $\mathcal{H} \subseteq \Upsilon$ ,  $\mathcal{H}$  may or may not be stated as a union of some equivalence classes of  $\sigma$ . If it is possible to express  $\mathcal{H}$  as the latter, then it is named definable; if not, it is an RS. If  $\mathcal{H}$  is an RS, then it can be approximated as:

$$\left. \begin{aligned} \sigma_*(\mathcal{H}) &= \{\varepsilon \in \Upsilon : [\varepsilon]_\sigma \subseteq \mathcal{H}\}, \\ \sigma^*(\mathcal{H}) &= \{\varepsilon \in \Upsilon : [\varepsilon]_\sigma \cap \mathcal{H} \neq \emptyset\}, \end{aligned} \right\} \quad (1)$$

which are referred to as lower and upper approximations of  $\mathcal{H}$ , respectively, where

$$[\varepsilon]_\sigma = \{\varepsilon' \in \Upsilon : (\varepsilon, \varepsilon') \in \sigma\}. \quad (2)$$

Further, the set

$$\text{Bnd}_\sigma(\mathcal{H}) = \sigma^*(\mathcal{H}) - \sigma_*(\mathcal{H}) \quad (3)$$

is called the boundary region of  $\mathcal{H}$  in  $\Upsilon$ . Obviously,  $\mathcal{H}$  is definable if  $\sigma_*(\mathcal{H}) = \sigma^*(\mathcal{H})$ .  $\mathcal{H}$  is a RS if  $\sigma_*(\mathcal{H}) \neq \sigma^*(\mathcal{H})$ .

**Definition 2.2** [1] A FS  $F$  on  $\Upsilon$  is a function  $F : \Upsilon \rightarrow [0, 1]$ . The value  $F(\varepsilon)$  of  $F$  for  $\varepsilon \in \Upsilon$  signifies the MG of  $\varepsilon$  in  $F$ .

**Definition 2.3** [16] A BFS  $\Psi$  over  $\Upsilon$  is postulated as:

$$\Psi = \left\{ \langle \varepsilon, \Psi^+(\varepsilon), \Psi^-(\varepsilon) \rangle : \varepsilon \in \Upsilon \right\}, \quad (4)$$

where  $\Psi^+ : \Upsilon \rightarrow [0, 1]$  and  $\Psi^- : \Upsilon \rightarrow [-1, 0]$  are said to be positive MG (PMG) and negative MG (NMG), respectively. The PMG reveals the satisfaction degree of an item  $\varepsilon$  to the property, and the NMG demonstrates the satisfaction degree of  $\varepsilon$  to some implicit counter-property related to a BFS  $\Psi$ .

From now on, the assembling of all BFSs over  $\Upsilon$  is denoted by  $\mathcal{BF}(\Upsilon)$ .

**Definition 2.4** [16] Let  $\Psi, \Gamma \in \mathcal{BF}(\Upsilon)$ . Then  $\forall \varepsilon \in \Upsilon$ ,

1.  $\Psi \subseteq \Gamma \Leftrightarrow \Psi^+(\varepsilon) \leq \Gamma^+(\varepsilon)$  and  $\Psi^-(\varepsilon) \geq \Gamma^-(\varepsilon)$ ;
2.  $\Psi = \Gamma \Leftrightarrow \Psi^+(\varepsilon) = \Gamma^+(\varepsilon)$  and  $\Psi^-(\varepsilon) = \Gamma^-(\varepsilon)$ ;
3.  $(\Psi \cap \Gamma)(\varepsilon) = \{\Psi^+(\varepsilon) \wedge \Gamma^+(\varepsilon), \Psi^-(\varepsilon) \vee \Gamma^-(\varepsilon)\}$ ;
4.  $(\Psi \cup \Gamma)(\varepsilon) = \{\Psi^+(\varepsilon) \vee \Gamma^+(\varepsilon), \Psi^-(\varepsilon) \wedge \Gamma^-(\varepsilon)\}$ ;
5.  $\Psi^c(\varepsilon) = \{1 - \Psi^+(\varepsilon), -1 - \Psi^-(\varepsilon)\}$ .

**Definition 2.5** [16] A BFR  $\mathbb{B}$  over  $\Upsilon$  is a BFS over  $\Upsilon \times \Upsilon$ . Thus, it may be formulated as:

$$\mathbb{B} = \left\{ \langle (x', y'), \mu_{\mathbb{B}}^+(x', y'), \mu_{\mathbb{B}}^-(x', y') \rangle : (x', y') \in \Upsilon \times \Upsilon \right\}, \quad (5)$$

where  $\mu_{\mathbb{B}}^+ : \Upsilon \times \Upsilon \rightarrow [0, 1]$  and  $\mu_{\mathbb{B}}^- : \Upsilon \times \Upsilon \rightarrow [-1, 0]$ .

Recently, Gul et al. [26] generated the paradigm of bipolar fuzzy preference relation (BFPR) along with its core features.

**Definition 2.6** [26] A BFPR  $\mathfrak{B}$  is a BFS over  $\Upsilon \times \Upsilon$ , which is expressed by its positive and negative MFs given as  $\mu_{\mathfrak{B}}^+ : \Upsilon \times \Upsilon \rightarrow [0, 1]$  and  $\mu_{\mathfrak{B}}^- : \Upsilon \times \Upsilon \rightarrow [-1, 0]$ . For  $\Upsilon = \{b_1, b_2, \dots, b_n\}$ , we can denote it as:

$$\mathfrak{B} = \left[ \langle a_{ij}, b_{ij} \rangle \right]_{n \times n} = \begin{matrix} & b_1 & b_2 & \dots & b_n \\ b_1 & \langle a_{11}, b_{11} \rangle & \langle a_{12}, b_{12} \rangle & \dots & \langle a_{1n}, b_{1n} \rangle \\ b_2 & \langle a_{21}, b_{21} \rangle & \langle a_{22}, b_{22} \rangle & \dots & \langle a_{2n}, b_{2n} \rangle \\ \vdots & \vdots & \ddots & \vdots & \\ b_n & \langle a_{n1}, b_{n1} \rangle & \langle a_{n2}, b_{n2} \rangle & \dots & \langle a_{nn}, b_{nn} \rangle \end{matrix},$$

where  $\langle a_{ij}, b_{ij} \rangle$  indicates the bipolar fuzzy preference degree (BFPD) of  $x_i$  over  $x_j$ ,  $a_{ij} \in [0, 1]$ ,  $b_{ij} \in [-1, 0]$ . Further,  $a_{ij}$  and  $b_{ij}$  fulfills the constraints,  $a_{ij} + a_{ji} = 1$ ,  $b_{ij} + b_{ji} = -1$ ,  $a_{ii} = 0.5$  and  $b_{ii} = -0.5 \forall i, j = 1, 2, \dots, n$ . Specially,

- $a_{ij} = 0.5, b_{ij} = -0.5$  indicates indifference between  $b_i$  and  $b_j$ ;
- $a_{ij} > 0.5, b_{ij} > -0.5$  reveals that  $b_i$  is better than  $b_j$ ;
- $a_{ij} < 0.5, b_{ij} < -0.5$  signifies that  $b_j$  is better than  $b_i$ ;
- $a_{ij} = 1, b_{ij} = 0$  suggests that  $b_i$  is absolutely better than  $b_j$ ;
- $a_{ij} = 0, b_{ij} = -1$  means  $b_j$  is absolutely better than  $b_i$ .

**Definition 2.7** [26] A BFPR  $\mathfrak{B} = \left[ \langle a_{ij}, b_{ij} \rangle \right]_{n \times n}$  is termed as an additive consistent, if  $\forall i, j, k \in \{1, 2, \dots, n\}$  the following conditions hold:

1.  $a_{ij} = a_{ik} - a_{jk} + 0.5$ ,
2.  $b_{ij} = b_{ik} - b_{jk} + 0.5$ .

**Definition 2.8** [26] Let  $\Upsilon = \{b_i : i = 1, 2, \dots, n\}$  be the universe of  $n$  items and  $C = \{C_k : k = 1, 2, \dots, m\}$  be a finite collection of  $m$  criteria. Let  $f : \Upsilon \times C \rightarrow [0, 1]$  be a fuzzy MF and  $g : \Upsilon \times C \rightarrow [-1, 0]$  be a fuzzy non-membership function. Then, we postulate the transfer functions to determine the BFPD of any two items  $b_i, b_j \in \Upsilon$  regarding the criterion  $C_k$  as follows:

$$a_{ij}^{C_k} = \frac{f(b_i, C_k) - f(b_j, C_k) + 1}{2}, \tag{6}$$

$$b_{ij}^{C_k} = \frac{g(b_j, C_k) - g(b_i, C_k) - 1}{2}. \tag{7}$$

For a BFPR  $\mathfrak{B}_{C_k}(b_i, b_j) = \left[ \langle a_{ij}^{C_k}, b_{ij}^{C_k} \rangle \right]_{n \times n}$ , the transfer functions (6) and (7) satisfied the following characteristics for all  $b_i, b_j, b_k \in \Upsilon$ :

1.  $a_{ii}^{C_k} = 0.5$  and  $b_{ii}^{C_k} = -0.5$ .
2.  $a_{ij}^{C_k} + a_{ji}^{C_k} = 1$  and  $b_{ij}^{C_k} + b_{ji}^{C_k} = -1$ .
3.  $a_{ij}^{C_k} + a_{j\ell}^{C_k} = a_{i\ell}^{C_k} + 0.5$  and  $b_{ij}^{C_k} + b_{j\ell}^{C_k} = b_{i\ell}^{C_k} - 0.5$ .

Table 1 illustrates a bipolar fuzzy matrix, where  $\Upsilon = \{b_i : i = 1, 2, \dots, 5\}$  and  $C = \{C_1, C_2\}$ . Based on criteria  $C_1$  and  $C_2$ , we can construct the BFPRs of alternative  $b_i$  to the alternative  $b_j$  ( $i, j =$

**Table 1**  
 Bipolar fuzzy matrix

$\Upsilon/C$	$C_1$	$C_2$
$x_1$	(0.5, - 0.25)	(0.8, - 0.7)
$x_2$	(0.25, - 0.8)	(0.9, - 0.4)
$x_3$	(0.33, - 0.25)	(0.75, - 0.4)
$x_4$	(0.65, - 0.6)	(0.3, - 0.75)
$x_5$	(1, - 0.5)	(0.4, - 0.35)

1, 2,  $\dots$ , 5) by employing formulas (6) and (7), we acquired:

$$\mathfrak{B}_{C_1}(b_i, b_j) = \begin{pmatrix} \langle 0.500, -0.500 \rangle & \langle 0.625, -0.775 \rangle & \langle 0.585, -0.500 \rangle & \langle 0.425, -0.675 \rangle & \langle 0.250, -0.625 \rangle \\ \langle 0.375, -0.225 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.460, -0.225 \rangle & \langle 0.300, -0.400 \rangle & \langle 0.125, -0.350 \rangle \\ \langle 0.415, -0.500 \rangle & \langle 0.540, -0.775 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.340, -0.675 \rangle & \langle 0.165, -0.625 \rangle \\ \langle 0.575, -0.325 \rangle & \langle 0.700, -0.600 \rangle & \langle 0.660, -0.325 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.325, -0.450 \rangle \\ \langle 0.750, -0.375 \rangle & \langle 0.875, -0.650 \rangle & \langle 0.835, -0.375 \rangle & \langle 0.675, -0.550 \rangle & \langle 0.500, -0.500 \rangle \end{pmatrix}, \tag{8}$$

$$\mathfrak{B}_{C_2}(b_i, b_j) = \begin{pmatrix} \langle 0.500, -0.500 \rangle & \langle 0.450, -0.350 \rangle & \langle 0.525, -0.350 \rangle & \langle 0.750, -0.525 \rangle & \langle 0.700, -0.325 \rangle \\ \langle 0.550, -0.650 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.575, -0.500 \rangle & \langle 0.800, -0.675 \rangle & \langle 0.750, -0.475 \rangle \\ \langle 0.475, -0.650 \rangle & \langle 0.425, -0.500 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.725, -0.675 \rangle & \langle 0.675, -0.475 \rangle \\ \langle 0.250, -0.475 \rangle & \langle 0.200, -0.325 \rangle & \langle 0.275, -0.325 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.450, -0.300 \rangle \\ \langle 0.300, -0.675 \rangle & \langle 0.250, -0.525 \rangle & \langle 0.325, -0.525 \rangle & \langle 0.550, -0.700 \rangle & \langle 0.500, -0.500 \rangle \end{pmatrix}. \tag{9}$$

**Definition 2.9** [26] The bipolar fuzzy preference classes (BFPCs) of an item  $x_i \in \Upsilon$  induced by a BFPR  $\mathfrak{B}_{C_k}(x_i, x_j) = \left[ \langle a_{ij}^{C_k}, b_{ij}^{C_k} \rangle \right]_{n \times n}$  is portrayed as:

$$[x_i]_{\mathfrak{B}_{C_k}} = \frac{\langle a_{i1}^{C_k}, b_{i1}^{C_k} \rangle}{x_1} + \frac{\langle a_{i2}^{C_k}, b_{i2}^{C_k} \rangle}{x_2} + \dots + \frac{\langle a_{in}^{C_k}, b_{in}^{C_k} \rangle}{x_n}. \tag{10}$$

The BFPR  $\mathfrak{B}_{C_k}$  produces a collection of bipolar fuzzy information granules from the universe, which constitutes the bipolar fuzzy preference granular structure described as follows:

$$P(\mathfrak{B}_{C_k}) = \left\{ [x_1]_{\mathfrak{B}_{C_k}}, [x_2]_{\mathfrak{B}_{C_k}}, \dots, [x_n]_{\mathfrak{B}_{C_k}} \right\}.$$

Gul et al. [26] implemented the formulas (6) and (7) to originate the conception of bipolar fuzzy preference  $\delta$ -neighborhood (BFP $\delta$ -nghd) in a BFP $\delta$ C-approximation space (BFP $\delta$ C-AS) and examined its associated features. Eventually, they design an innovative BFP $\delta$ C-BFRS variant via BFP $\delta$ -nghd.

**Definition 2.10** [26] Presume  $\Upsilon$  be a non-void finite universe and  $P(\mathfrak{B}_{C_k})$  be a bipolar fuzzy preference granular structure. Then for every  $\delta = \langle \alpha, \beta \rangle \in (0, 1] \times [-1, 0)$ , we name  $P(\mathfrak{B}_{C_k})$  a BFP $\delta$ C of  $\Upsilon$ , when

$$\left( \bigcup_{i,j=1}^n a_{ij}^{C_k} \right)(x) \geq \alpha \text{ and } \left( \bigcap_{i,j=1}^n b_{ij}^{C_k} \right)(x) \leq \beta, \forall x \in \Upsilon. \quad (11)$$

Furthermore, the pair  $(\mathcal{U}, P(\mathfrak{B}_{C_k}))$  is termed a BFP $\delta$ C-AS.

**Definition 2.11** [26] Let  $(\mathcal{U}, P(\mathfrak{B}_{C_k}))$  be a BFP $\delta$ C-AS. For each  $x \in \Upsilon$ , we characterize the BFP $\delta$ -nghd  $\mathfrak{N}_x^\delta$  of  $x$  as:

$$\mathfrak{N}_x^\delta = \langle \mathfrak{N}_x^\alpha, \mathfrak{N}_x^\beta \rangle, \quad (12)$$

where,

$$\mathfrak{N}_x^\alpha = \bigwedge \left\{ [x_i]_{\mathfrak{B}_{C_k}} : a_{ij}^{C_k} \geq \alpha \right\}, \quad (13)$$

and

$$\mathfrak{N}_x^\beta = \bigvee \left\{ [x_i]_{\mathfrak{B}_{C_k}} : b_{ij}^{C_k} \leq \beta \right\}. \quad (14)$$

**Definition 2.12** [26] Assume that  $(\Upsilon, P(\mathfrak{B}_{C_k}))$  be a BFP $\delta$ C-AS and  $\delta = \langle \alpha, \beta \rangle \in (0, 1] \times [-1, 0)$ . The BFP $\delta$ C lower and upper approximations of a BFS  $\Psi = \langle \Psi^+, \Psi^- \rangle$  in  $\Upsilon$  regarding  $(\mathcal{U}, P(\mathfrak{B}_{C_k}))$  are respectively postulated as:

$$\left. \begin{aligned} \underline{BF}_C(\Psi) &= \left\langle \underline{(\Psi^+)}(x)_C, \underline{(\Psi^-)}(x)_C \right\rangle, \\ \overline{BF}_C(\Psi) &= \left\langle \overline{(\Psi^+)}(x)_C, \overline{(\Psi^-)}(x)_C \right\rangle, \end{aligned} \right\} \quad (15)$$

where,

$$\left. \begin{aligned} \underline{(\Psi^+)}(x)_C &= \bigwedge_{y \in \Upsilon} \left\{ \left( 1 - {}^C \mathfrak{N}_x^\alpha(y) \right) \vee \Psi^+(y) \right\}, \\ \underline{(\Psi^-)}(x)_C &= \bigvee_{y \in \Upsilon} \left\{ {}^C \mathfrak{N}_x^\beta(y) \wedge \Psi^-(y) \right\}, \\ \overline{(\Psi^+)}(x)_C &= \bigvee_{y \in \Upsilon} \left\{ {}^C \mathfrak{N}_x^\alpha(y) \wedge \Psi^+(y) \right\}, \\ \overline{(\Psi^-)}(x)_C &= \bigwedge_{y \in \Upsilon} \left\{ \left( -1 - {}^e \mathfrak{N}_x^\beta(y) \right) \vee \Psi^-(y) \right\}, \text{ for every } x \in \Upsilon. \end{aligned} \right\} \quad (16)$$

If  $\underline{BF}_C(\Psi) \neq \overline{BF}_C(\Psi)$ , then  $\Psi$  is named BFP $\delta$ C-BFRS; else,  $\Psi$  is termed a bipolar fuzzy definable.

### 3. An Integrated MCDM Using BFP $\delta$ C-BFRS Model and VIKOR

In the current segment, we introduced a novel MCDM strategy by unifying the BFP $\delta$ C- BFRS model and the VIKOR scheme.

Consider a collection  $\Upsilon = \{x_i : i = 1, 2, \dots, n\}$  of  $n$  items and  $C = \{C_k : k = 1, 2, \dots, m\}$  be a collection of  $m$  criteria that are decided by a decision expert. The associated weight vector of

all criteria is signified by  $\mathfrak{W} = (\hbar_1, \hbar_2, \dots, \hbar_m)^T$  such that  $0 \leq \hbar_j \leq 1$  and  $\sum_{j=1}^m \hbar_j = 1$ . Let the BFS  $\Psi = \{\langle x, \Psi^+(x), \Psi^-(x) \rangle : x \in \Upsilon\}$  over  $\Upsilon$  be the description of all items by the expert. Let  $\mathbb{E}$  signify a finite set of the domain for the information functions  $f(x_\ell, C_k)$  and  $g(x_\ell, C_k)$ , where  $f(x_\ell, C_k) \in [0, 1]$  stands for the PMG of  $x_\ell$  w.r.t.  $C_k$  given by the expert and  $g(x_\ell, C_k) \in [-1, 0]$  stands for the NMG of  $x_\ell$  w.r.t.  $C_k$  given by the expert. The BFPC  $[x_i]_{\mathfrak{B}_{C_k}}(x_j)$  stands for the efficacy value of  $x_j$ 's w.r.t.  $C_k$ . For a critical value  $\delta = \langle \alpha, \beta \rangle \in (0, 1] \times [-1, 0)$ , let for each  $x_i \in \Upsilon$ , there is at least one criteria  $C_k \in C$  such that the efficacy value of  $x_j$  for the criteria  $C_k$  is not less than  $\alpha$  and greater than  $\beta$ , and  $P(\mathfrak{B}_{C_k})$  is a  $BFP\delta C$  of  $\Upsilon$ . Then the  $BFP\delta$ -nghd  $\mathfrak{N}_{x_j}^\delta$  of  $x_j$  regarding  $C_k$  is a BFS described as:

$$C_k \mathfrak{N}_{x_j}^\delta(x_r) = \left[ \bigwedge \{a_{ij}^{C_k} : a_{ij}^{C_k} \geq \alpha\}, \bigvee \{b_{ij}^{C_k} : b_{ij}^{C_k} \leq \beta\} \right](x_r); r = 1, 2, \dots, n, \quad (17)$$

which yields the minimum of all efficacy values for each  $x_r$  w.r.t.  $C_k$ .

If  $\Psi = \{\langle x_i, \Psi^+(x_i), \Psi^-(x_i) \rangle : x_i \in \Upsilon\} \in \mathcal{BF}(\Upsilon)$  represent PMG and NMG of each  $x_i \in \Upsilon$  given by the expert. The decision-making for the MCDM problem is how to select an optimal alternative among all. We denote the bipolar fuzzy information system  $(\Upsilon, C, \mathfrak{W}, \mathbb{E})$ .

### 3.1 Decision-making Methodology

The recommended method has subsequent steps.

**Step 1:** Assume that the decision-maker furnish his/her assessment of the alternatives  $x_i$  regarding each criterion  $C_k$  by considering bipolar fuzzy numbers as  $x_{ik} = \langle \Psi_k^+(x_i), \Psi_k^-(x_i) \rangle$ , where the PMG  $\Psi_k^+(x_i)$  signifies the satisfaction degree of alternatives  $x_i$  to the criteria  $C_k$  and the NMG  $\Psi_k^-(x_i)$  denotes the satisfaction degree of the alternatives  $x_i$  to some counter property of the criteria  $C_k$ . Formally, a bipolar fuzzy MCDM problem can be represented by an  $n \times m$  matrix as follows:

$$\mathfrak{D} = \left[ \langle \Psi_k^+(x_i), \Psi_k^-(x_i) \rangle \right]_{n \times m}. \quad (18)$$

**Step 2:** Compute BFPRs  $\mathfrak{B}_{C_k}$ ;  $k = 1, 2, \dots, m$  using formulas (6) and (7), respectively.

**Step 3:** Determine the BFPCs  $[x_i]_{\mathfrak{B}_{C_k}}$  of alternatives  $x_i$  induced by each BFPR.

**Step 4:** Construct the  $BFP\delta$ -nghds  $C_k \mathfrak{N}_{x_i}^\delta = \langle C_k \mathfrak{N}_{x_i}^\alpha, C_k \mathfrak{N}_{x_i}^\beta \rangle$  of alternatives  $x_i$  corresponding to each criteria  $C_k$  by using Definition 2.11.

**Step 5:** All the individual  $BFP\delta$ -nghds can be transformed into a final aggregated  $BFP\delta$ -nghd, which is given as:

$$\widetilde{\mathcal{N}}_{x_i}^\delta = \langle \widetilde{\mathcal{N}}_{x_i}^\alpha, \widetilde{\mathcal{N}}_{x_i}^\beta \rangle, \quad (19)$$

where

$$\widetilde{\mathcal{N}}_{x_i}^\alpha = \frac{\sum_{k=1}^m C_k \mathfrak{N}_{x_i}^\alpha}{m}, \quad (20)$$

and

$$\widetilde{\mathcal{N}}_{x_i}^\beta = \frac{\sum_{k=1}^m C_k \mathfrak{N}_{x_i}^\beta}{m}. \quad (21)$$

**Step 6:** According to Eq. (18), the best value  $f_k^\spadesuit = \langle f_k^{\spadesuit+}, f_k^{\spadesuit-} \rangle$  and the worst value  $f_k^\star = \langle f_k^{\star+}, f_k^{\star-} \rangle$  can be determine against each criteria  $C_k$  as follows:

$$f_k^\spadesuit = \langle f_k^{\spadesuit+}, f_k^{\spadesuit-} \rangle = \left\{ \left\langle \bigvee_{i=1}^n \Psi_k^+(x_i), \bigwedge_{i=1}^n \Psi_k^-(x_i) \right\rangle : k = 1, 2, \dots, m \right\}, \quad (22)$$

$$f_k^\star = \langle f_k^{\star+}, f_k^{\star-} \rangle = \left\{ \left\langle \bigwedge_{i=1}^n \Psi_k^+(x_i), \bigvee_{i=1}^n \Psi_k^-(x_i) \right\rangle : k = 1, 2, \dots, m \right\}. \quad (23)$$

**Step 7:** Calculate the maximum group utility value  $S_i = (S_i^P, S_i^N)$  and the minimum individual regret value  $\mathcal{R}_i = (\mathcal{R}_i^P, \mathcal{R}_i^N)$  as follows:

$$S_i^P = \sum_{k=1}^n \bar{h}_k \frac{d(f_k^{\spadesuit+}, \Psi_k^+(x_i))}{d(f_k^{\spadesuit+}, f_k^{\star+})}, \quad (24)$$

$$S_i^N = \sum_{k=1}^n \bar{h}_k \frac{d(\Psi_k^-(x_i), f_k^{\star-})}{d(f_k^{\star-}, f_k^{\spadesuit-})} \quad (25)$$

and

$$\mathcal{R}_i^P = \bigvee_i \left( \bar{h}_k \frac{d(f_k^{\spadesuit+}, \Psi_k^+(x_i))}{d(f_k^{\spadesuit+}, f_k^{\star+})} \right), \quad (26)$$

$$\mathcal{R}_i^N = \bigwedge_i \left( \bar{h}_k \frac{d(\Psi_k^-(x_i), f_k^{\star-})}{d(f_k^{\star-}, f_k^{\spadesuit-})} \right). \quad (27)$$

**Step 8:** Further, by means of Definition 2.12, find the  $BFP\delta C$  lower and upper approximations of utility measure and regret measure w.r.t. final aggregated  $BFP\delta$ -nghd  $\widetilde{\mathcal{N}}_{x_i}^\delta$ , which are respectively given as:

$$\left. \begin{aligned} \underline{(S_i^P)}(x) &= \bigwedge_{y \in \Upsilon} \left\{ (1 - \widetilde{\mathcal{N}}_{x_i}^\alpha(y)) \vee S_i^P(y) \right\}, \\ \underline{(S_i^N)}(x) &= \bigvee_{y \in \Upsilon} \left\{ \widetilde{\mathcal{N}}_{x_i}^\beta(y) \wedge S_i^N(y) \right\}, \\ \overline{(S_i^P)}(x) &= \bigvee_{y \in \Upsilon} \left\{ \widetilde{\mathcal{N}}_{x_i}^\alpha(y) \wedge S_i^P(y) \right\}, \\ \overline{(S_i^N)}(x) &= \bigwedge_{y \in \Upsilon} \left\{ (-1 - \widetilde{\mathcal{N}}_{x_i}^\beta(y)) \vee S_i^N(y) \right\}, \text{ for every } x \in \Upsilon. \end{aligned} \right\} \quad (28)$$

And

$$\left. \begin{aligned} \underline{(\mathcal{R}_i^P)}(x) &= \bigwedge_{y \in \Upsilon} \left\{ \left( 1 - \widetilde{\mathcal{N}}_{x_i}^\alpha(y) \right) \vee \mathcal{R}_i^P(y) \right\}, \\ \underline{(\mathcal{R}_i^N)}(x) &= \bigvee_{y \in \Upsilon} \left\{ \widetilde{\mathcal{N}}_{x_i}^\beta(y) \wedge \mathcal{R}_i^N(y) \right\}, \\ \overline{(\mathcal{R}_i^P)}(x) &= \bigvee_{y \in \Upsilon} \left\{ \widetilde{\mathcal{N}}_{x_i}^\alpha(y) \wedge \mathcal{R}_i^P(y) \right\}, \\ \overline{(\mathcal{R}_i^N)}(x) &= \bigwedge_{y \in \Upsilon} \left\{ \left( -1 - \widetilde{\mathcal{N}}_{x_i}^\beta(y) \right) \vee \mathcal{R}_i^N(y) \right\}, \text{ for every } x \in \Upsilon. \end{aligned} \right\} \quad (29)$$

**Step 9:** Next, by utilizing the *BFPδC* lower and upper approximations of  $\mathcal{S}_i$  and  $\mathcal{R}_i$ , we find the lower and upper approximation vectors as follows:

$$\underline{\mathcal{S}}_i = \underline{\mathcal{S}}_i^P \oplus \underline{\mathcal{S}}_i^N, \quad (30)$$

$$\overline{\mathcal{S}}_i = \overline{\mathcal{S}}_i^P \oplus \overline{\mathcal{S}}_i^N, \quad (31)$$

$$\underline{\mathcal{R}}_i = \underline{\mathcal{R}}_i^P \oplus \underline{\mathcal{R}}_i^N, \quad (32)$$

$$\overline{\mathcal{R}}_i = \overline{\mathcal{R}}_i^P \oplus \overline{\mathcal{R}}_i^N, \quad (33)$$

**Step 10:** Find the values of  $\mathcal{S}'_i$  and  $\mathcal{R}'_i$  by using the following formulas:

$$\mathcal{S}'_i = \underline{\mathcal{S}}_i \oplus \overline{\mathcal{S}}_i, \quad (34)$$

$$\mathcal{R}'_i = \underline{\mathcal{R}}_i \oplus \overline{\mathcal{R}}_i. \quad (35)$$

Afterward, we determine the compromise  $\mathcal{Q}_i$  for each alternatives by using the following formula:

$$\mathcal{Q}_i = \gamma \left( \frac{\mathcal{S}'_i - \mathcal{S}^*}{\mathcal{S}^- - \mathcal{S}^*} \right) + (1 - \gamma) \left( \frac{\mathcal{R}'_i - \mathcal{R}^*}{\mathcal{R}^- - \mathcal{R}^*} \right), \quad (36)$$

where

$$\mathcal{S}^* = \bigwedge_{i=1}^n \mathcal{S}'_i, \quad \mathcal{S}^- = \bigvee_{i=1}^n \mathcal{S}'_i, \quad \mathcal{R}^* = \bigwedge_{i=1}^n \mathcal{R}'_i, \quad \mathcal{R}^- = \bigvee_{i=1}^n \mathcal{R}'_i;$$

and  $\gamma$  is served as a weight for the strategy of maximum group utility, while  $1 - \gamma$  is the weight of individual regret and its value falls in  $[0, 1]$ . These strategies could be compromised (consensus) by  $\gamma = 0.5$ . The compromise solution can be chosen by majority  $\gamma > 0.5$  and veto  $\gamma < 0.5$ .

The items are ranked by arranging the values of  $\mathcal{S}'_i$ ,  $\mathcal{R}'_i$ , and  $\mathcal{Q}_i$  in ascending order. The outcomes are three ranking lists w.r.t. the values of  $\mathcal{S}'_i$ ,  $\mathcal{R}'_i$ , and  $\mathcal{Q}_i$ , which are further used to propose the compromise solution of alternatives. The term  $\mathcal{Q}_i$  is the separation measure of alternative  $x_i$  from the optimal alternative, which shows that the minimum value of  $\mathcal{Q}$  indicates the optimal alternative.

Lastly, we choose the optimal alternative  $x_1$  as a compromise solution if the following two constraints are fulfilled:

### Condition I (Tolerable advantage)

The first condition is illustrated as:

$$Q(x^{(2)}) - Q(x^{(1)}) \geq \frac{1}{n-1}, \quad (37)$$

where  $x^{(1)}$  and  $x^{(2)}$  are high ranked two alternatives in  $Q_i$  and  $n$  is the number of alternatives.

### Condition II (Tolerable stability)

In this condition the alternative  $x^{(1)}$  should also best ranked alternative in  $S'_i$  and  $\mathcal{R}'_i$ .

If the above two conditions do not hold together, then there are many compromise solutions, which are attained as follows:

1. If the only *condition II* is not fulfilled then the set of compromise solutions comprises alternatives  $x^{(1)}$  and  $x^{(2)}$ .
2. If the only *condition I* is not satisfied then the compromise solution consists of alternatives  $x^{(1)}, x^{(2)}, \dots, x^{(N)}$ , where  $x^{(N)}$  is determine by the relation:

$$Q(x^{(N)}) - Q(x^{(1)}) < \frac{1}{n-1}; \text{ for maximum } N. \quad (38)$$

In light of the above discussion, the procedure of the devised VIKOR method under the BFP $\delta$ C-BFRS model is portrayed in the following algorithm. Moreover, Figure 1 depicts the graphical illustration of the recommended MCDM mechanism.

---

#### Algorithm 1: An algorithm for the MCDM problem

---

Input: Bipolar fuzzy information system  $(\Upsilon, C, \mathfrak{M}, \mathcal{E})$ .

Output: A ranking result of all alternatives.

Step 1: Determine  $\mathfrak{B}_{C_k}; k = 1, 2, \dots, m$  using formulation exhibited in Eqs. (6) and (7).

Step 2: Compute  $[x_i]_{\mathfrak{B}_{C_k}}$  of  $x_i$  regarding  $C_k$ .

Step 3: Compute  ${}^{C_k}N_{x_i}^\delta = \langle {}^{C_k}N_{x_i}^\alpha, {}^{C_k}N_{x_i}^\beta \rangle$  of  $x_i$  regarding  $C_k$ .

Step 4: Determine aggregated BFP $\delta$ -nghd  $\widetilde{N}_{x_i}^\delta$  using Eqs. (19), (20) and (21).

Step 5: Compute the best and the wrest values against each criterion  $C_k$  according to Eqs. (22) and (23).

Step 6: Calculate  $S_i = (S_i^P, S_i^N)$  and  $\mathcal{R}_i = (\mathcal{R}_i^P, \mathcal{R}_i^N)$  according to Eqs. (24), (25), (26) and (27).

Step 7: Find the BFP $\delta$ C lower and upper approximations of  $S_i$  and  $\mathcal{R}_i$  w.r.t.  $\widetilde{N}_{x_i}^\delta$  using Eqs. (28) and (29).

Step 8: Calculate the lower and upper approximation vectors.

Step 9: Compute the values of  $S'_i, \mathcal{R}'_i$  and  $Q_i$  according to Eq. (36).

Step 10: Rank the items in terms of  $S'_i, \mathcal{R}'_i$  and  $Q_i$ .

---

## 4. Application of the Proposed VIKOR Approach

In this part, a real-world decision-making issue employing bipolar fuzzy information has been utilized to analyze the developed approach.

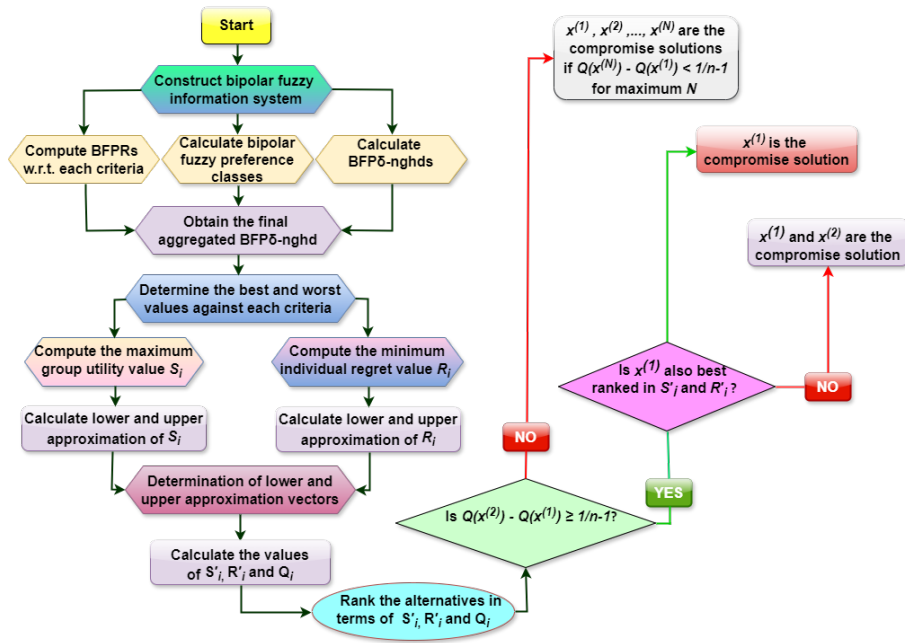


Fig. 1. Flowchart portrayal of the projected of the approach

Imagine a decision-maker who is having trouble choosing between the five smartphones that are on display for sale in a store. Since it is well known that each feature influences the price and usefulness of the desired smartphone, let's presume that he is concentrating on the subsequent features (criteria) to choose the finest smartphone: colour, memory, elegancy, and camera zoom. Therefore, the attributes listed above stand in for the requirements in our MCDM issue, and these smartphones represent the alternatives.

Let  $C = \{C_1, C_2, C_3, C_4\}$  be the set of concerned features, and  $\Upsilon = \{x_1, x_2, x_3, x_4, x_5\}$  be the collection of concerned smartphones. All the criteria weights are provided as follows:  $\mathfrak{W} = (0.25, 0.3, 0.25, 0.2)^T$ . Table 2 shows the alternatives' ratings according to the criteria.

Table 2  
 Bipolar fuzzy information system

$\Upsilon/C$	$C_1$	$C_2$	$C_3$	$C_4$
$x_1$	(0.5, - 0.25)	(0.8, - 0.7)	(0.3, - 0.1)	(0.6, - 0.6)
$x_2$	(0.2, - 0.8)	(0.9, - 0.4)	(0.6, - 0.3)	(0.55, - 0.5)
$x_3$	(0.33, - 0.25)	(0.75, - 0.4)	(0.25, - 0.7)	(0.3, - 0.1)
$x_4$	(0.65, - 0.6)	(0.3, - 0.75)	(0.8, - 0.35)	(0.65, - 0.7)
$x_5$	(1, - 0.5)	(0.4, - 0.35)	(0.2, - 0.6)	(0.25, - 0.65)

The following computation phases are displayed to help choose the best smartphone:

**Step 1:** Based on criteria  $C_1, C_2, C_3, C_4$  and according to formulas (6) and (7) to compute the BFPs of

alternative  $x_i$  to the alternative  $x_j (i, j = 1, 2, \dots, 5)$ , we get:

$$\mathfrak{B}_{C_1}(x_i, x_j) = \begin{pmatrix} \langle 0.500, -0.500 \rangle & \langle 0.650, -0.775 \rangle & \langle 0.585, -0.500 \rangle & \langle 0.425, -0.675 \rangle & \langle 0.250, -0.625 \rangle \\ \langle 0.350, -0.225 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.435, -0.225 \rangle & \langle 0.275, -0.400 \rangle & \langle 0.100, -0.350 \rangle \\ \langle 0.415, -0.500 \rangle & \langle 0.565, -0.775 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.340, -0.675 \rangle & \langle 0.165, -0.625 \rangle \\ \langle 0.575, -0.325 \rangle & \langle 0.725, -0.600 \rangle & \langle 0.660, -0.325 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.325, -0.450 \rangle \\ \langle 0.750, -0.375 \rangle & \langle 0.900, -0.650 \rangle & \langle 0.835, -0.375 \rangle & \langle 0.675, -0.550 \rangle & \langle 0.500, -0.500 \rangle \end{pmatrix}, \tag{39}$$

$$\mathfrak{B}_{C_2}(x_i, x_j) = \begin{pmatrix} \langle 0.500, -0.500 \rangle & \langle 0.450, -0.350 \rangle & \langle 0.525, -0.350 \rangle & \langle 0.750, -0.525 \rangle & \langle 0.700, -0.325 \rangle \\ \langle 0.550, -0.650 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.575, -0.500 \rangle & \langle 0.800, -0.675 \rangle & \langle 0.750, -0.475 \rangle \\ \langle 0.475, -0.650 \rangle & \langle 0.425, -0.500 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.725, -0.675 \rangle & \langle 0.675, -0.475 \rangle \\ \langle 0.250, -0.475 \rangle & \langle 0.200, -0.325 \rangle & \langle 0.275, -0.325 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.450, -0.300 \rangle \\ \langle 0.300, -0.675 \rangle & \langle 0.250, -0.525 \rangle & \langle 0.325, -0.525 \rangle & \langle 0.550, -0.700 \rangle & \langle 0.500, -0.500 \rangle \end{pmatrix}, \tag{40}$$

$$\mathfrak{B}_{C_3}(x_i, x_j) = \begin{pmatrix} \langle 0.500, -0.500 \rangle & \langle 0.350, -0.600 \rangle & \langle 0.525, -0.800 \rangle & \langle 0.250, -0.625 \rangle & \langle 0.550, -0.750 \rangle \\ \langle 0.650, -0.400 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.675, -0.700 \rangle & \langle 0.400, -0.525 \rangle & \langle 0.700, -0.650 \rangle \\ \langle 0.475, -0.200 \rangle & \langle 0.325, -0.300 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.225, -0.325 \rangle & \langle 0.525, -0.450 \rangle \\ \langle 0.750, -0.375 \rangle & \langle 0.600, -0.475 \rangle & \langle 0.775, -0.675 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.800, -0.625 \rangle \\ \langle 0.450, -0.250 \rangle & \langle 0.300, -0.350 \rangle & \langle 0.475, -0.550 \rangle & \langle 0.200, -0.375 \rangle & \langle 0.500, -0.500 \rangle \end{pmatrix}, \tag{41}$$

$$\mathfrak{B}_{C_4}(x_i, x_j) = \begin{pmatrix} \langle 0.500, -0.500 \rangle & \langle 0.525, -0.450 \rangle & \langle 0.650, -0.250 \rangle & \langle 0.475, -0.550 \rangle & \langle 0.675, -0.525 \rangle \\ \langle 0.475, -0.550 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.625, -0.300 \rangle & \langle 0.450, -0.600 \rangle & \langle 0.650, -0.575 \rangle \\ \langle 0.350, -0.750 \rangle & \langle 0.375, -0.700 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.325, -0.800 \rangle & \langle 0.525, -0.775 \rangle \\ \langle 0.525, -0.450 \rangle & \langle 0.550, -0.400 \rangle & \langle 0.675, -0.200 \rangle & \langle 0.500, -0.500 \rangle & \langle 0.700, -0.475 \rangle \\ \langle 0.325, -0.475 \rangle & \langle 0.350, -0.425 \rangle & \langle 0.475, -0.225 \rangle & \langle 0.300, -0.525 \rangle & \langle 0.500, -0.500 \rangle \end{pmatrix}. \tag{42}$$

**Step 2:** The BFPCs  $[x_i]_{\mathfrak{B}_{C_1}}, [x_i]_{\mathfrak{B}_{C_2}}, [x_i]_{\mathfrak{B}_{C_3}}$  and  $[x_i]_{\mathfrak{B}_{C_4}}$  are respectively listed in Tables 3, 4, 5 and 6.

From Tables 3, 4, 5 and 6, we can observe that  $P(\mathfrak{B}_{\mathfrak{C}_k}) = \{[x_i]_{\mathfrak{B}_{\mathfrak{C}_k}} : i = 1, 2, \dots, 5, k = 1, 2, 3, 4\}$  is a BFP $\delta$ C of  $\Upsilon$  ( $\delta = \langle 0.500, -0.500 \rangle$ ).

**Step 3:** Let  $\delta = \langle \alpha, \beta \rangle = \langle 0.500, -0.500 \rangle$  be the critical value. Then the elements  $C_k \mathfrak{N}_{x_i}^\delta = \langle C_k \mathfrak{N}_{x_i}^\alpha, C_k \mathfrak{N}_{x_i}^\beta \rangle$  ( $i = 1, 2, \dots, 5, k = 1, 2, 3, 4$ ) are displayed in Tables 7, 8, 9 and 10.

**Step 4:** The aggregated BFP $\delta$ -nghd  $\widetilde{\mathfrak{N}}_{x_i}^\delta$  is given in Table 11 as follows:

**Step 5:** The best and the worst values against each criteria  $C_k$  can be evaluated according to Eqs. (22) and (23), respectively, which are shown in Table 12.

**Table 3**  
 The BFPSc  $[x_i]_{\mathfrak{B}_{C_1}}$

	$[x_1]_{\mathfrak{B}_{C_1}}$	$[x_2]_{\mathfrak{B}_{C_1}}$	$[x_3]_{\mathfrak{B}_{C_1}}$	$[x_4]_{\mathfrak{B}_{C_1}}$	$[x_5]_{\mathfrak{B}_{C_1}}$
$x_1$	$\langle 0.500, -0.500 \rangle$	$\langle 0.350, -0.225 \rangle$	$\langle 0.415, -0.500 \rangle$	$\langle 0.575, -0.325 \rangle$	$\langle 0.750, -0.375 \rangle$
$x_2$	$\langle 0.650, -0.775 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.565, -0.775 \rangle$	$\langle 0.725, -0.600 \rangle$	$\langle 0.900, -0.650 \rangle$
$x_3$	$\langle 0.585, -0.500 \rangle$	$\langle 0.435, -0.225 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.660, -0.325 \rangle$	$\langle 0.835, -0.375 \rangle$
$x_4$	$\langle 0.425, -0.675 \rangle$	$\langle 0.275, -0.400 \rangle$	$\langle 0.340, -0.675 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.675, -0.550 \rangle$
$x_5$	$\langle 0.250, -0.625 \rangle$	$\langle 0.100, -0.350 \rangle$	$\langle 0.165, -0.625 \rangle$	$\langle 0.325, -0.450 \rangle$	$\langle 0.500, -0.500 \rangle$

**Table 4**  
 The BFPCs  $[x_i]_{\mathfrak{B}_{C_2}}$

	$[x_1]_{\mathfrak{B}_{C_2}}$	$[x_2]_{\mathfrak{B}_{C_2}}$	$[x_3]_{\mathfrak{B}_{C_2}}$	$[x_4]_{\mathfrak{B}_{C_2}}$	$[x_5]_{\mathfrak{B}_{C_2}}$
$x_1$	$\langle 0.500, -0.500 \rangle$	$\langle 0.550, -0.650 \rangle$	$\langle 0.475, -0.650 \rangle$	$\langle 0.250, -0.475 \rangle$	$\langle 0.300, -0.675 \rangle$
$x_2$	$\langle 0.450, -0.350 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.425, -0.500 \rangle$	$\langle 0.200, -0.325 \rangle$	$\langle 0.250, -0.525 \rangle$
$x_3$	$\langle 0.525, -0.350 \rangle$	$\langle 0.575, -0.500 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.275, -0.325 \rangle$	$\langle 0.325, -0.525 \rangle$
$x_4$	$\langle 0.750, -0.525 \rangle$	$\langle 0.800, -0.675 \rangle$	$\langle 0.725, -0.675 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.550, -0.700 \rangle$
$x_5$	$\langle 0.700, -0.325 \rangle$	$\langle 0.750, -0.475 \rangle$	$\langle 0.675, -0.475 \rangle$	$\langle 0.450, -0.300 \rangle$	$\langle 0.500, -0.500 \rangle$

**Table 5**  
 The BFPCs  $[x_i]_{\mathfrak{B}_{C_3}}$

	$[x_1]_{\mathfrak{B}_{C_3}}$	$[x_2]_{\mathfrak{B}_{C_3}}$	$[x_3]_{\mathfrak{B}_{C_3}}$	$[x_4]_{\mathfrak{B}_{C_3}}$	$[x_5]_{\mathfrak{B}_{C_3}}$
$x_1$	$\langle 0.500, -0.500 \rangle$	$\langle 0.650, -0.400 \rangle$	$\langle 0.475, -0.200 \rangle$	$\langle 0.750, -0.375 \rangle$	$\langle 0.450, -0.250 \rangle$
$x_2$	$\langle 0.350, -0.600 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.325, -0.300 \rangle$	$\langle 0.600, -0.475 \rangle$	$\langle 0.300, -0.350 \rangle$
$x_3$	$\langle 0.525, -0.800 \rangle$	$\langle 0.675, -0.700 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.775, -0.675 \rangle$	$\langle 0.475, -0.550 \rangle$
$x_4$	$\langle 0.250, -0.625 \rangle$	$\langle 0.400, -0.525 \rangle$	$\langle 0.225, -0.325 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.200, -0.375 \rangle$
$x_5$	$\langle 0.550, -0.750 \rangle$	$\langle 0.700, -0.650 \rangle$	$\langle 0.525, -0.450 \rangle$	$\langle 0.800, -0.625 \rangle$	$\langle 0.500, -0.500 \rangle$

**Table 6**  
 The BFPCs  $[x_i]_{\mathfrak{B}_{C_4}}$

	$[x_1]_{\mathfrak{B}_{C_4}}$	$[x_2]_{\mathfrak{B}_{C_4}}$	$[x_3]_{\mathfrak{B}_{C_4}}$	$[x_4]_{\mathfrak{B}_{C_4}}$	$[x_5]_{\mathfrak{B}_{C_4}}$
$x_1$	$\langle 0.500, -0.500 \rangle$	$\langle 0.475, -0.550 \rangle$	$\langle 0.350, -0.750 \rangle$	$\langle 0.525, -0.450 \rangle$	$\langle 0.325, -0.475 \rangle$
$x_2$	$\langle 0.525, -0.450 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.375, -0.700 \rangle$	$\langle 0.550, -0.400 \rangle$	$\langle 0.350, -0.425 \rangle$
$x_3$	$\langle 0.650, -0.250 \rangle$	$\langle 0.625, -0.300 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.675, -0.200 \rangle$	$\langle 0.475, -0.225 \rangle$
$x_4$	$\langle 0.475, -0.550 \rangle$	$\langle 0.450, -0.600 \rangle$	$\langle 0.325, -0.800 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.300, -0.525 \rangle$
$x_5$	$\langle 0.675, -0.525 \rangle$	$\langle 0.650, -0.575 \rangle$	$\langle 0.525, -0.775 \rangle$	$\langle 0.700, -0.475 \rangle$	$\langle 0.500, -0.500 \rangle$

**Step 6:** The values of utility measure  $S_i = (S_i^+, S_i^-)$  and regret measure  $\mathcal{R}_i = (\mathcal{R}_i^+, \mathcal{R}_i^-)$  are evaluated by using Eqs. (24), (25), (26) and (27), respectively, and the outcomes are displayed in Table 13.

**Step 7:** Using Eqs. (28) and (29), the BFP $\delta$ C lower and upper approximations of  $S_i$  and  $\mathcal{R}_i$  regarding  $\widetilde{\mathcal{N}}_{x_i}^\delta$  are

**Table 7**  
 The  $BFP\delta$ -nghd  $C_1N_{x_i}^\delta$

	$C_1N_{x_1}^\delta$	$C_1N_{x_2}^\delta$	$C_1N_{x_3}^\delta$	$C_1N_{x_4}^\delta$	$C_1N_{x_5}^\delta$
$x_1$	$\langle 0.500, -0.500 \rangle$	$\langle 0.350, -0.225 \rangle$	$\langle 0.415, -0.500 \rangle$	$\langle 0.575, -0.325 \rangle$	$\langle 0.750, -0.375 \rangle$
$x_2$	$\langle 0.650, -0.775 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.565, -0.775 \rangle$	$\langle 0.725, -0.600 \rangle$	$\langle 0.900, -0.650 \rangle$
$x_3$	$\langle 0.585, -0.500 \rangle$	$\langle 0.435, -0.225 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.660, -0.325 \rangle$	$\langle 0.835, -0.375 \rangle$
$x_4$	$\langle 0.425, -0.675 \rangle$	$\langle 0.275, -0.400 \rangle$	$\langle 0.340, -0.675 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.675, -0.550 \rangle$
$x_5$	$\langle 0.250, -0.625 \rangle$	$\langle 0.100, -0.350 \rangle$	$\langle 0.165, -0.625 \rangle$	$\langle 0.325, -0.450 \rangle$	$\langle 0.500, -0.500 \rangle$

**Table 8**  
 The  $BFP\delta$ -nghd  $C_2N_{x_i}^\delta$

	$C_2N_{x_1}^\delta$	$C_2N_{x_2}^\delta$	$C_2N_{x_3}^\delta$	$C_2N_{x_4}^\delta$	$C_2N_{x_5}^\delta$
$x_1$	$\langle 0.500, -0.500 \rangle$	$\langle 0.550, -0.650 \rangle$	$\langle 0.475, -0.650 \rangle$	$\langle 0.250, -0.475 \rangle$	$\langle 0.300, -0.675 \rangle$
$x_2$	$\langle 0.450, -0.350 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.425, -0.500 \rangle$	$\langle 0.200, -0.325 \rangle$	$\langle 0.250, -0.525 \rangle$
$x_3$	$\langle 0.525, -0.350 \rangle$	$\langle 0.575, -0.500 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.275, -0.325 \rangle$	$\langle 0.325, -0.525 \rangle$
$x_4$	$\langle 0.750, -0.525 \rangle$	$\langle 0.800, -0.675 \rangle$	$\langle 0.725, -0.675 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.550, -0.700 \rangle$
$x_5$	$\langle 0.700, -0.325 \rangle$	$\langle 0.750, -0.475 \rangle$	$\langle 0.675, -0.475 \rangle$	$\langle 0.450, -0.300 \rangle$	$\langle 0.500, -0.500 \rangle$

**Table 9**  
 The  $BFP\delta$ -nghd  $C_3N_{x_i}^\delta$

	$C_3N_{x_1}^\delta$	$C_3N_{x_2}^\delta$	$C_3N_{x_3}^\delta$	$C_3N_{x_4}^\delta$	$C_3N_{x_5}^\delta$
$x_1$	$\langle 0.500, -0.500 \rangle$	$\langle 0.650, -0.400 \rangle$	$\langle 0.475, -0.200 \rangle$	$\langle 0.750, -0.375 \rangle$	$\langle 0.450, -0.250 \rangle$
$x_2$	$\langle 0.350, -0.600 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.325, -0.300 \rangle$	$\langle 0.600, -0.475 \rangle$	$\langle 0.300, -0.350 \rangle$
$x_3$	$\langle 0.525, -0.800 \rangle$	$\langle 0.675, -0.700 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.775, -0.675 \rangle$	$\langle 0.475, -0.550 \rangle$
$x_4$	$\langle 0.250, -0.625 \rangle$	$\langle 0.400, -0.525 \rangle$	$\langle 0.225, -0.325 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.200, -0.375 \rangle$
$x_5$	$\langle 0.550, -0.750 \rangle$	$\langle 0.700, -0.650 \rangle$	$\langle 0.525, -0.450 \rangle$	$\langle 0.800, -0.625 \rangle$	$\langle 0.500, -0.500 \rangle$

**Table 10**  
 The  $BFP\delta$ -nghd  $C_4N_{x_i}^\delta$

	$C_4N_{x_1}^\delta$	$C_4N_{x_2}^\delta$	$C_4N_{x_3}^\delta$	$C_4N_{x_4}^\delta$	$C_4N_{x_5}^\delta$
$x_1$	$\langle 0.500, -0.500 \rangle$	$\langle 0.475, -0.550 \rangle$	$\langle 0.350, -0.750 \rangle$	$\langle 0.525, -0.450 \rangle$	$\langle 0.325, -0.475 \rangle$
$x_2$	$\langle 0.525, -0.450 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.375, -0.700 \rangle$	$\langle 0.550, -0.400 \rangle$	$\langle 0.350, -0.425 \rangle$
$x_3$	$\langle 0.650, -0.250 \rangle$	$\langle 0.625, -0.300 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.675, -0.200 \rangle$	$\langle 0.475, -0.225 \rangle$
$x_4$	$\langle 0.475, -0.550 \rangle$	$\langle 0.450, -0.600 \rangle$	$\langle 0.325, -0.800 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.300, -0.525 \rangle$
$x_5$	$\langle 0.675, -0.525 \rangle$	$\langle 0.650, -0.575 \rangle$	$\langle 0.525, -0.775 \rangle$	$\langle 0.700, -0.475 \rangle$	$\langle 0.500, -0.500 \rangle$

respectively given in Table 14.

**Step 8:** We find the lower and upper approximation vectors using Eqs. (30 - 33) and the results are listed in Table 15 as follows:

**Table 11**  
 The aggregated BFP $\delta$ -nghd  $\widetilde{\mathcal{N}}_{x_i}^\delta$

	$\widetilde{\mathcal{N}}_{x_1}^\delta$	$\widetilde{\mathcal{N}}_{x_2}^\delta$	$\widetilde{\mathcal{N}}_{x_3}^\delta$	$\widetilde{\mathcal{N}}_{x_4}^\delta$	$\widetilde{\mathcal{N}}_{x_5}^\delta$
$x_1$	$\langle 0.500, -0.500 \rangle$	$\langle 0.506, -0.456 \rangle$	$\langle 0.429, -0.525 \rangle$	$\langle 0.525, -0.406 \rangle$	$\langle 0.456, -0.444 \rangle$
$x_2$	$\langle 0.494, -0.544 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.423, -0.569 \rangle$	$\langle 0.519, -0.450 \rangle$	$\langle 0.450, -0.488 \rangle$
$x_3$	$\langle 0.571, -0.475 \rangle$	$\langle 0.578, -0.431 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.596, -0.381 \rangle$	$\langle 0.528, -0.419 \rangle$
$x_4$	$\langle 0.475, -0.594 \rangle$	$\langle 0.481, -0.550 \rangle$	$\langle 0.404, -0.619 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.431, -0.538 \rangle$
$x_5$	$\langle 0.544, -0.556 \rangle$	$\langle 0.550, -0.513 \rangle$	$\langle 0.473, -0.581 \rangle$	$\langle 0.569, -0.463 \rangle$	$\langle 0.500, -0.500 \rangle$

**Table 12**  
 The values of  $f_k^+$  and  $f_k^-$

	$C_1$	$C_2$	$C_3$	$C_4$
$f_k^+$	$\langle 1, -0.8 \rangle$	$\langle 0.9, -0.75 \rangle$	$\langle 0.8, -0.7 \rangle$	$\langle 0.65, -0.7 \rangle$
$f_k^-$	$\langle 0.2, -0.25 \rangle$	$\langle 0.3, -0.35 \rangle$	$\langle 0.2, -0.1 \rangle$	$\langle 0.25, -0.1 \rangle$

**Table 13**  
 The values of  $\mathcal{S}_i$  and  $\mathcal{R}_i$

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$
$\mathcal{S}_i$	$\langle 0.4396, -0.4667 \rangle$	$\langle 0.3833, -0.5095 \rangle$	$\langle 0.6885, -0.2929 \rangle$	$\langle 0.4094, -0.8061 \rangle$	$\langle 0.7000, -0.5053 \rangle$
$\mathcal{R}_i$	$\langle 0.1562, -0.3000 \rangle$	$\langle 0.2500, -0.2500 \rangle$	$\langle 0.2094, -0.2500 \rangle$	$\langle 0.3000, -0.3429 \rangle$	$\langle 0.2500, -0.2083 \rangle$

**Table 14**  
 The lower and upper approximations of  $\mathcal{S}_i$  and  $\mathcal{R}_i$

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$
$(\underline{\mathcal{S}}_i^P, \underline{\mathcal{S}}_i^N)$	$\langle 0.425, -0.475 \rangle$	$\langle 0.494, -0.431 \rangle$	$\langle 0.571, -0.500 \rangle$	$\langle 0.475, -0.381 \rangle$	$\langle 0.544, -0.419 \rangle$
$(\overline{\mathcal{S}}_i^P, \overline{\mathcal{S}}_i^N)$	$\langle 0.571, -0.4667 \rangle$	$\langle 0.578, -0.500 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.596, -0.5095 \rangle$	$\langle 0.528, -0.5095 \rangle$
$(\underline{\mathcal{R}}_i^P, \underline{\mathcal{R}}_i^N)$	$\langle 0.429, -0.475 \rangle$	$\langle 0.422, -0.431 \rangle$	$\langle 0.500, -0.500 \rangle$	$\langle 0.404, -0.381 \rangle$	$\langle 0.472, -0.419 \rangle$
$(\overline{\mathcal{R}}_i^P, \overline{\mathcal{R}}_i^N)$	$\langle 0.300, -0.3429 \rangle$	$\langle 0.300, -0.3429 \rangle$	$\langle 0.300, -0.3429 \rangle$	$\langle 0.300, -0.3429 \rangle$	$\langle 0.300, -0.3429 \rangle$

**Table 15**  
 Values of  $\underline{\mathcal{S}}_i, \overline{\mathcal{S}}_i, \underline{\mathcal{R}}_i$  and  $\overline{\mathcal{R}}_i$

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$
$\underline{\mathcal{S}}_i$	-0.05	0.063	0.071	0.094	0.125
$\overline{\mathcal{S}}_i$	0.1043	0.078	0	0.0865	0.0185
$\underline{\mathcal{R}}_i$	-0.046	-0.009	0	0.023	0.053
$\overline{\mathcal{R}}_i$	-0.0429	-0.0429	-0.0429	-0.0429	-0.0429

**Step 9:** Finally, by employing Eqs. (34), (35) and (36), the outputs of  $\mathcal{S}'_i, \mathcal{R}'_i$  and  $\mathcal{Q}_i$  for  $\gamma = 0.5$  are determined in Table 16. By taking the ascending order of  $\mathcal{S}'_i, \mathcal{R}'_i$  and  $\mathcal{Q}_i$ , we acquired three distinct ranking outcomes of five alternatives which are displayed in Table 16. These rankings are pictorially depicted in Figure 2.

In light of Table 16, it becomes evident that  $x_1$  is the optimal alternative for all three ranking lists.

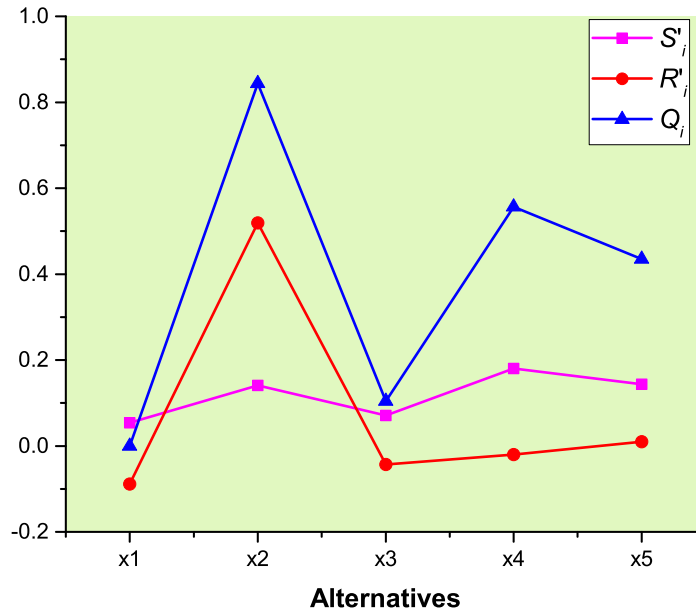
**Table 16**  
 Values of  $S'_i$ ,  $R'_i$  and  $Q_i$

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	Ranking
$S'_i$	0.0543	0.141	0.071	0.1805	0.1435	$x_1 \succ x_3 \succ x_2 \succ x_5 \succ x_4$
$R'_i$	-0.0889	0.519	-0.0429	-0.0199	0.0101	$x_1 \succ x_3 \succ x_4 \succ x_5 \succ x_2$
$Q_i$	0	0.8435	0.1040	0.5568	0.4348	$x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_2$

But,

$$Q(x^{(2)}) - Q(x^{(1)}) = Q(x_3) - Q(x_1) = 0.1040 < \frac{1}{5-1} = 0.25.$$

Hence, the proposed algorithm's first condition (tolerable advantage) is not justified. So,  $\{x_1, x_3\}$  is a set of compromise solutions.



**Fig. 2.** Ranking of alternatives

## 5. Comparative study and Discussion

The benefits of the suggested methodology are examined in this section, along with a comparison with existing methods. We compare our devised scheme with some prevalent approaches in the literature including, Malik and Shabir [24], Wei *et al.* [27], Jana *et al.* [28], Gul [29], and Gul *et al.* [30]. These comparison outcomes are displayed in Table 17.

According to 17, we can see that the ranking outcomes of alternatives of the other approaches are different, mainly due to alterations in the decision-making context. Yet, the optimal choice is identical. This fact is common in decision analysis. Generally, decision-makers can allocate various inputs of  $\alpha$  and  $\beta$  according to their preferences and real demands.

**Table 17**  
 Comparison with some existing methods

Methods	Ranking of alternatives	Optimal alternative
Malik and Shabir [24]	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5$	$x_1$
Wei et al. [27]	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5$	$x_1$
Jana et al. [28]	$x_1 \succ x_3 \succ x_4 \succ x_2 \succ x_5$	$x_1$
Gul [29]	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5$	$x_1$
Gul et al. [30]	$x_1 \approx x_4 \succ x_3 \succ x_5 \succ x_2$	$x_1, x_4$
Our proposed approach	$x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_2$	$x_1$

### 5.1 Merits of the devised strategy

To illustrate the uniqueness and superiority of our suggested strategy, we go over the drawbacks of current approaches and how the recommended framework addresses these issues.

1. MCDM issues with bipolar fuzzy data are studied using a variety of BFR-based decision-making techniques. But, not every MADM issue can be described by a BFR. Because of this, we provide an approach based on the BFP $\delta$ C-BFRS variant for tackling MCDM issues using bipolar fuzzy information.
2. Comparing our proposed method to the approaches described in [1, 10-15], we find that these approaches are unable to adequately reflect bipolarity in the decision procedure, which is a crucial aspect of human perception and behaviour.
3. Fuzzy decision-making strategies are substantially adopted to address issues with merely one-sided data; i.e., objects are ranked via the PMG. By using fuzzy structure in decision-making, we are unable to offer details about the dissatisfaction degree of alternatives regarding various criteria. Therefore, we advocate a BFP $\delta$ C-BFRSs framework to rank the items.

### 5.2 Limitations

The framed scheme highly relies on the adequate input of the parameters  $\alpha$  and  $\beta$ . Selecting the best items is difficult and frequently arbitrary. In practice, various inputs of  $\alpha$  and  $\beta$  may provide varied ranking outcomes, making the technique less reliable.

## 6. Conclusions

FSs and RSs have several practical applications and are efficient mathematical techniques for circumventing uncertainty. Meanwhile, BFSs address both fuzziness and bipolarity simultaneously. In the MCDM technique, the alternatives are compared against each other based on their relative performance to each other. Furthermore, to ascertain the relative importance of each criterion, the VIKOR technique necessitates a comparison of the criteria. The best compromise solution is determined by the highest ranking of the alternatives. In this script, we established an innovative hybrid VIKOR method within the context of the BFP $\delta$ C-BFRSs. The proficiency and applicability of the framed methodology have been highlighted using a numerical example. Finally, we conducted a comparison

of the devised methodology with some other prevailing approaches and analyzed how our framed approach is superior to existing ones.

In the future, we extend our to more generalized frameworks including covering-based  $(\alpha, \beta)$ -multi-granulation bipolar FRS model [30], and m-polar FSs [31].

### Acknowledgement

This research was not funded by any grant.

### Conflicts of Interest

The authors declare no conflicts of interest.

### References

- [1] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, 8(3), 338–353. [https://doi.org/10.1016/S0019-9958\(65\)90241-X](https://doi.org/10.1016/S0019-9958(65)90241-X)
- [2] Zhang, S., Hou, Y., Zhang, S., & Zhang, M. (2017). Fuzzy control model and simulation for non-linear supply chain system with lead times. *Complexity*, 2017(1), 2017634. <https://doi.org/10.1155/2017/2017634>
- [3] Zhang, S., Li, S., Zhang, S., & Zhang, M. (2017). Decision of lead-time compression and stable operation of supply chain. *Complexity*, 2017(1), 7436764. <https://doi.org/10.1155/2017/7436764>
- [4] Zhang, S., Zhang, C., Zhang, S., & Zhang, M. (2018). Discrete switched model and fuzzy robust control of dynamic supply chain network. *Complexity*, 2018(1), 3495096. <https://doi.org/10.1155/2018/3495096>
- [5] Zhang, S., Zhang, P., & Zhang, M. (2019). Fuzzy emergency model and robust emergency strategy of supply chain system under random supply disruptions. *Complexity*, 2019(1), 3092514. <https://doi.org/10.1155/2019/3092514>
- [6] Pawlak, Z. (1982). Rough sets. *International journal of computer & information sciences*, 11, 341–356. <https://doi.org/10.1007/BF01001956>
- [7] Pawlak, Z., & Skowron, A. (2007). Rudiments of rough sets. *Information sciences*, 177(1), 3–27. <https://doi.org/10.1016/j.ins.2006.06.003>
- [8] Skowron, A., & Stepaniuk, J. (1996). Tolerance approximation spaces. *Fundamenta Informaticae*, 27(2-3), 245–253. <https://doi.org/10.3233/FI-1996-272311>
- [9] Yao, Y. (1998). Relational interpretations of neighborhood operators and rough set approximation operators. *Information sciences*, 111(1-4), 239–259. [https://doi.org/10.1016/S0020-0255\(98\)10006-3](https://doi.org/10.1016/S0020-0255(98)10006-3)
- [10] Dubois, D., & Prade, H. (1992). Putting rough sets and fuzzy sets together. In *Putting rough sets and fuzzy sets together. in intelligent decision support: Handbook of applications and advances of the rough sets theory* (pp. 203–232). dordrecht: Springer netherlands. [https://doi.org/10.1007/978-94-015-7975-9\\_14](https://doi.org/10.1007/978-94-015-7975-9_14)
- [11] Dubois, D., & Prade, H. (1990). Rough fuzzy sets and fuzzy rough sets. *International Journal of General System*, 17(2-3), 191–209. <https://doi.org/10.1080/03081079008935107>
- [12] Greco, S., Matarazzo, B., & Slowinski, R. (2002). Rough approximation by dominance relations. *International journal of intelligent systems*, 17(2), 153–171. <https://doi.org/10.1002/int.10014>
- [13] Ziarko, W. (1993). Variable precision rough set model. *Journal of computer and system sciences*, 46(1), 39–59. [https://doi.org/10.1016/0022-0000\(93\)90048-2](https://doi.org/10.1016/0022-0000(93)90048-2)

- [14] Yang, B. (2022). Fuzzy covering-based rough set on two different universes and its application. *Artificial Intelligence Review*, 55(6), 4717–4753. <https://doi.org/10.1007/s10462-021-10115-y>
- [15] Hu, Y.-C. (2016). Pattern classification using grey tolerance rough sets. *Kybernetes*, 45(2), 266–281. <https://doi.org/10.1108/K-04-2015-0105>
- [16] Zhang, W.-R. (1994). Bipolar fuzzy sets and relations: A computational framework for cognitive modeling and multiagent decision analysis. *Bipolar fuzzy sets and relations: A computational framework for cognitive modeling and multiagent decision analysis. NAFIPS/IFIS/NASA'94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intelligence*, (pp. 305–309). <https://doi.org/10.1109/IJCF.1994.375115>
- [17] Han, Y., Shi, P., & Chen, S. (2015). Bipolar-valued rough fuzzy set and its applications to the decision information system. *IEEE Transactions on Fuzzy Systems*, 23(6), 2358–2370. <https://doi.org/10.1109/TFUZZ.2015.2423707>
- [18] Yang, H.-L., Li, S.-G., Wang, S., & Wang, J. (2012). Bipolar fuzzy rough set model on two different universes and its application. *Knowledge-Based Systems*, 35, 94–101. <https://doi.org/10.1016/j.knosys.2012.01.001>
- [19] Wei, G., Wei, C., & Gao, H. (2018). Multiple attribute decision making with interval-valued bipolar fuzzy information and their application to emerging technology commercialization evaluation. *IEEE Access*, 6, 60930–60955. <https://doi.org/10.1109/ACCESS.2018.2875261>
- [20] Ali, G., Akram, M., & Alcantud, J. C. R. (2020). Attributes reductions of bipolar fuzzy relation decision systems. *Neural computing and applications*, 32(14), 10051–10071. <https://doi.org/10.1007/s00521-019-04536-8>
- [21] Gul, R., & Shabir, M. (2020). Roughness of a set by  $(\alpha, \beta)$ -indiscernibility of bipolar fuzzy relation. *Computational and Applied Mathematics*, 39(3), 160. <https://doi.org/10.1007/s40314-020-01174-y>
- [22] Han, Y., Chen, S., & Shen, X. (2022). Fuzzy rough set with inconsistent bipolarity information in two universes and its applications. *Soft Computing*, 26(19), 9775–9784. <https://doi.org/10.1007/s00500-022-07356-6>
- [23] Jana, C., & Pal, M. (2021). Extended bipolar fuzzy edas approach for multi-criteria group decision-making process. *Computational and Applied Mathematics*, 40, 1–15. <https://doi.org/10.1007/s40314-020-01403-4>
- [24] Malik, N., & Shabir, M. (2019). A consensus model based on rough bipolar fuzzy approximations. *Journal of Intelligent & Fuzzy Systems*, 36(4), 3461–3470. <https://doi.org/10.2333/JIFS-181223>
- [25] Luo, J., & Hu, M. (2023). A bipolar three-way decision model and its application in analyzing incomplete data. *International Journal of Approximate Reasoning*, 152, 94–123. <https://doi.org/10.1016/j.ijar.2022.10.011>
- [26] Gul, R., Shabir, M., & Naeem, M. (2023). A comprehensive study on  $(\alpha, \beta)$ -bipolar fuzzified rough set model based on bipolar fuzzy preference relation and corresponding decision-making applications. *Computational and Applied Mathematics*, 42(7), 310. <https://doi.org/10.1007/s40314-023-02430-7>
- [27] Wei, G., Alsaadi, F. E., Hayat, T., & Alsaedi, A. (2018). Bipolar fuzzy hamacher aggregation operators in multiple attribute decision making. *International Journal of Fuzzy Systems*, 20, 1–12. <https://doi.org/10.1007/s40815-017-0338-6>
- [28] Jana, C., Pal, M., & Wang, J.-Q. (2019). Bipolar fuzzy dombi aggregation operators and its application in multiple-attribute decision-making process. *Journal of Ambient Intelligence and Humanized Computing*, 10, 3533–3549. <https://doi.org/10.1007/s12652-018-1076-9>

- [29] Gul, Z. (2015). Some bipolar fuzzy aggregations operators and their applications in multicriteria group decision making (doctoral dissertation, m. phil thesis).
- [30] Gul, R., Shabir, M., & Al-Kenani, A. N. (2024). Covering-based  $(\alpha, \beta)$ -multi-granulation bipolar fuzzy rough set model under bipolar fuzzy preference relation with decision-making applications. *Complex Intelligent System*, 10, 4351–4372. <https://doi.org/10.1007/s40747-024-01371-w>
- [31] Chen, J., Li, S., Ma, S., & Wang, X. (2014). M-polar fuzzy sets: An extension of bipolar fuzzy sets. *The scientific world journal*, 2014(1), 416530. <https://doi.org/10.1155/2014/416530>