SCIENTIFIC OASIS

Spectrum of Operational Research

Journal homepage: www.sor-journal.org
ISSN: 3042-1470

Fairly Aggregation Operators Based on Complex p, q-Rung Orthopair Fuzzy Sets and Their Application in Decision-Making Problems

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ARTICLE INFO

Article history:
Received 15 July 2024
Received in revised form 13 October 2024
Accepted 3 November 2024
Available online 11 November 2024

Keywords:
Decision-making; Complex p, q-Rung orthopair fuzzy sets; Fairly aggregation operators; Fuzzy sets.

ABSTRACT

The decision-making technique is used to evaluate the best optimal option among a collection of finite alternatives. Further, the technique of complex p, q-rung orthopair fuzzy (CPQROF) sets is very reliable and dominant due to the parameters "p" and "q". In contrast, the technique of simple q-rung orthopair fuzzy sets is a special case of the CPQROF set. The major theme of this article is to expose the novel theory of fairly operational laws based on CPQROF numbers (CPQROFNs). Further, we evaluate the weighted fairly aggregation operators based on CPQROF information, called the CPQROF weighted fairly averaging (CPQROFWFA) operator and the CPQROF ordered weighted fairly averaging (CPQROFOWFA) operator. Some properties are also discussed for the above operators. Additionally, we validate the system of multi-attribute decision-making (MADM) difficulties based on the initiated operators. Finally, we associate our suggested ranking consequences with approximately prevailing techniques to demonstrate the proposed theory's sovereignty and validity.

1. Introduction

To derive the best optimal is a very complex and challenging task for scholars, where the technique of MADM technique is very suitable for evaluating the best decision among the collection of alternatives based on some suitable criteria. Further, during the decision-making procedure, we lost a lot of information because of crisp information, whereas in the case of a crisp set, we have just two options such as zero and one, but in many cases, the crisp information has not worked effectively because of limited options. For this, Zadeh [1] proposed the fuzzy set (FSs). Further, Atanassov [2,3] introduced the intuitionistic FS (IFSs), because of the complications, where the FSs have only truth grade, but in many situations, we have needed two types of information, called truth grade and falsity grade with a condition that is the sum of both grades contained in unit interval, which is the part of IFSs. Moreover, Yager [4] proposed the Pythagorean FSs (PFSs) with a new condition that is the sum of the squares of both grades contained in the unit interval, where the IFSs are the special case of the PFSs. Additionally, in 2016, Yager [5] again proposed a new theory, called q-rung orthopair FSs

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https://doi.org/10.31181/sor21202514

(QROFSs) with a new technique, called $0 \leq \mathbb{S}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}}}(\ddot{x}) + \mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}}}(\ddot{x}) \leq 1, \mathbb{O}_{\mathbb{CQ}} \geq 1$, where $\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}$ represents the truth grade and $\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}$ represents the falsity of grades. Many applications of the IFSs, PFSs, and QROFSs have been proposed by different scholars, for instance, aggregation operators [6,7] and decision-making problems [8,9].

Fixing a complex number instead of a real number in the range of FSs is very complex and complicated because of vagueness and ambiguity in genuine life problems. In 2002, Ramot *et al.*, [10] proposed the complex FSs (CFSs), where the truth grade in CFSs is defined from a universal set to complex-valued unit interval, such as $[0,1] + i[0,1]$. Further, Alkouri and Salleh [11] initiated the complex IFSs (CIFSs) with a truth grade and falsity grade, such as$0 \leq \mathbb{S}^{1}_{\mathbb{Z}_{\mathbb{RP}}}(\ddot{x}) + \mathbb{T}^{1}_{\mathbb{Z}_{\mathbb{RP}}}(\ddot{x}) \leq 1$ and $0 \leq \mathbb{S}^{1}_{\mathbb{Z}_{\mathbb{IP}}}(\ddot{x}) + \mathbb{T}^{1}_{\mathbb{Z}_{\mathbb{IP}}}(\ddot{x}) \leq 1$. Moreover, Ullah *et al.,* [12] proposed the complex PFSs (CPFSs), the structure of CPFSs is the same as CIFSs but the condition of CPFSs is more modified as compared to CIFSs, such as the sum of the square of the real part (also for imaginary parts) of both grades contained in the unit interval. The technique of CPFSs is the modified version of the PFSs, CIFSs, IFSs, and FSs to cope with indeterminate and untrustworthy info in genuine life problems. Further, Liu *et al.*, [13,14] proposed the complex QROFSs (CQROFSs) with a novel condition such as $0 \leq \mathbb{S}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}}}(\ddot{x}) + \mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}}}(\ddot{x}) \leq 1$ and $0 \leq \mathbb{S}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}}}(\ddot{x}) + \mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}}}(\ddot{x}) \leq 1, \mathbb{O}_{\mathbb{CQ}} \geq 1$. Many applications of CIFSs, CPFSs, and CQROFSs have been proposed by different scholars, for instance, aggregation operators [15,16], hybrid operators [17,18], decision-making [19,20], and many others [21,22].

Various scholars have modified it in the shape of different structures, for instance, the technique of p, q-rung orthopair fuzzy set (PQROFS) was initiated by Ibrahim and Alshammari [23] with a new condition $0 \leq \mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}}}(\ddot{x}) + \mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}}}(\ddot{x}) \leq 1$, where the structure of QROFSs and PQROFSs are the same, but their condition are different, if $\mathfrak{Q}_{\mathbb{CQ}} = \mathbb{O}_{\mathbb{CQ}} = 1$, then the idea of PQROFSs is converted to QROFSs (a special case of the PQROFSs) [24]. Further, the novel theory of CPQROFSs was initiated by Ibrahim [25]. The key influence of this object is planned below:

i. To interpretation the original theory of fairly effective rules based on CPQROFNs.
ii. To evaluate the weighted fairly aggregation operators based on CPQROF information, called CPQROFWFA operator and CPQROFOWFA operator.
iii. To propose some properties for the above operators, called idempotency, monotonicity, and boundedness.
iv. To demonstrate the technique of MADM problems based on initiated operators.
v. To compare our proposed ranking results with some existing techniques to show the supremacy and validity of the proposed theory.

The article is arranged in the following shape: In Section 2, we revised the technique of CPQROFSs and their operational laws. In Section 3, we exposed the novel theory of fairly operational laws based on CPQROFNs. In Section 4, we evaluated the CPQROFWFA operator and CPQROFOWFA operator. Some properties are also discussed for the above operators, called idempotency, monotonicity, and boundedness. In Section 5, we demonstrated the technique of MADM problems based on initiated operators. In Section 6, we compared our proposed ranking results with some existing techniques to show the supremacy and validity of the proposed theory. Some valuable remarks are discussed in Section 7.

2. Preliminaries

In this section, we proposed the idea of CPQROFS and its operational laws.

Definition 1 [25]: The mathematical shape:

$$\mathbb{Z}_{\mathbb{CQ}} = \left\{\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{CQ}}}(\ddot{x}), \mathbb{T}_{\mathbb{Z}_{\mathbb{CQ}}}(\ddot{x})\right) : \ddot{x} \in \ddot{X}\right\}$$

called CPQROFS, noticed that $\mathbb{S}_{\mathbb{Z}_{\mathbb{CQ}}}(\mathfrak{x}) = \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}(\mathfrak{x})\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}(\mathfrak{x})\right)}$ and $\mathbb{T}_{\mathbb{Z}_{\mathbb{CQ}}}(\mathfrak{x}) = \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}(\mathfrak{x})\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}(\mathfrak{x})\right)}$ represents the supporting and supporting against grade with $0 \leq \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}(\mathfrak{x}) + \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}(\mathfrak{x}) \leq 1$ and $0 \leq \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}(\mathfrak{x}) + \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}(\mathfrak{x}) \leq 1, \mathbb{O}_{\mathbb{CQ}} \geq 1$. further, the terms

$$\mathcal{J}_{\mathbb{Z}_{\mathbb{CQ}}}(\mathfrak{x}) = \mathcal{J}_{\mathbb{Z}_{\mathbb{RP}}}(\mathfrak{x})\mathfrak{E}^{i2\pi\left(\mathcal{J}_{\mathbb{Z}_{\mathbb{IP}}}\right)} = \begin{pmatrix} 1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}(\mathfrak{x}) - \\ \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}(\mathfrak{x}) \end{pmatrix}^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \mathfrak{E}^{i2\pi\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}(\mathfrak{x})-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}(\mathfrak{x})\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}$$

shows the indeterminacy. Finally, the technique: $\mathbb{Z}_{\mathbb{CQ}} = \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}\right)}, \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}\right)}\right)$, called CPQROFS.

Definition 2 [25]: The idea of score and accuracy functions are listed below:

$$\mathfrak{S}_{\mathbb{CQ}}(\mathbb{Z}_{\mathbb{CQ}}) = \frac{1}{2}\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}} + \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right), \mathfrak{S}_{\mathbb{CQ}}(\mathbb{Z}_{\mathbb{CQ}}) \in [-1,1]$$

$$\mathfrak{H}_{\mathbb{CQ}}(\mathbb{Z}_{\mathbb{CQ}}) = \frac{1}{2}\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}} + \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}} + \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}} + \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right), \mathfrak{H}_{\mathbb{CQ}}(\mathbb{Z}_{\mathbb{CQ}}) \in [0,1]$$

With some properties:

i. If $\mathfrak{S}_{\mathbb{CQ}}(\mathbb{Z}_{\mathbb{CQ}-1}) > \mathfrak{S}_{\mathbb{CQ}}(\mathbb{Z}_{\mathbb{CQ}-2}) \Rightarrow \mathbb{Z}_{\mathbb{CQ}-1} > \mathbb{Z}_{\mathbb{CQ}-2}$;

vi. If $\mathfrak{S}_{\mathbb{CQ}}(\mathbb{Z}_{\mathbb{CQ}-1}) < \mathfrak{S}_{\mathbb{CQ}}(\mathbb{Z}_{\mathbb{CQ}-2}) \Rightarrow \mathbb{Z}_{\mathbb{CQ}-1} < \mathbb{Z}_{\mathbb{CQ}-2}$;

vii. If $\mathfrak{S}_{\mathbb{CQ}}(\mathbb{Z}_{\mathbb{CQ}-1}) = \mathfrak{S}_{\mathbb{CQ}}(\mathbb{Z}_{\mathbb{CQ}-2}) \Rightarrow$;

   a. If $\mathfrak{H}_{\mathbb{CQ}}(\mathbb{Z}_{\mathbb{CQ}-1}) > \mathfrak{H}_{\mathbb{CQ}}(\mathbb{Z}_{\mathbb{CQ}-2}) \Rightarrow \mathbb{Z}_{\mathbb{CQ}-1} > \mathbb{Z}_{\mathbb{CQ}-2}$;

   b. If $\mathfrak{H}_{\mathbb{CQ}}(\mathbb{Z}_{\mathbb{CQ}-1}) < \mathfrak{H}_{\mathbb{CQ}}(\mathbb{Z}_{\mathbb{CQ}-2}) \Rightarrow \mathbb{Z}_{\mathbb{CQ}-1} < \mathbb{Z}_{\mathbb{CQ}-2}$.

Definition 3 [25]: We discussed some operational laws:

$$\mathbb{Z}_{\mathbb{CQ}-1} \oplus \mathbb{Z}_{\mathbb{CQ}-2} = \begin{pmatrix} \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}} + \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{\frac{1}{\mathfrak{Q}_{\mathbb{CQ}}}} \mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{\frac{1}{\mathfrak{Q}_{\mathbb{CQ}}}}}, \\ \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}\right)} \end{pmatrix}$$

$$\mathbb{Z}_{\mathbb{CQ}-1} \otimes \mathbb{Z}_{\mathbb{CQ}-2} = \begin{pmatrix} \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}\right)}, \\ \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}} + \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}}} \mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}}}} \end{pmatrix}$$

$$\rho_{\mathbb{CQ}}\mathbb{Z}_{\mathbb{CQ}-1} = \left(\left(1 - \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)^{\frac{1}{\mathfrak{Q}_{\mathbb{CQ}}}} \mathfrak{E}^{i2\pi\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)^{\frac{1}{\mathfrak{Q}_{\mathbb{CQ}}}}}, \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\rho_{\mathbb{CQ}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\rho_{\mathbb{CQ}}}\right)}\right)$$

$$\mathbb{Z}_{\mathbb{CQ}-1}^{\rho_{\mathbb{CQ}}} = \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\rho_{\mathbb{CQ}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\rho_{\mathbb{CQ}}}\right)}, \left(1 - \left(1 - \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}}} \mathfrak{E}^{i2\pi\left(1-\left(1-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}}}}\right)$$

Definition 4 [25]: The CPQROFWA operator id is derived below:

$$\mathbb{CQ}ROFWA\left(\mathbb{Z}_{\mathbb{CQ}-1}, \mathbb{Z}_{\mathbb{CQ}-2}, \dots, \mathbb{Z}_{\mathbb{CQ}-\tilde{k}}\right)$$

$$= \begin{pmatrix} \left(1 - \prod_{\mathfrak{L}=1}^{\tilde{k}}\left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)^{\frac{1}{\mathfrak{Q}_{\mathbb{CQ}}}} \mathfrak{E}^{i2\pi\left(1-\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)^{\frac{1}{\mathfrak{Q}_{\mathbb{CQ}}}}}, \\ \prod_{\mathfrak{L}=1}^{\tilde{k}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\;\;w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\mathfrak{E}^{i2\pi\left(\prod_{\mathfrak{L}=1}^{\tilde{k}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\;\;w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)} \end{pmatrix}$$

Definition 5 [25]: The CPQROFOWA operator is derived below:

$$\mathbb{CQ}ROFOWA\left(\mathbb{Z}_{\mathbb{CQ}-1},\mathbb{Z}_{\mathbb{CQ}-2},\dots,\mathbb{Z}_{\mathbb{CQ}-\tilde{k}}\right)$$
$$=\left(\begin{array}{c}\left(1-\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-O(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)^{\frac{1}{\mathfrak{Q}_{\mathbb{CQ}}}}\mathfrak{E}^{i2\pi\left(1-\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-O(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)^{\frac{1}{\mathfrak{Q}_{\mathbb{CQ}}}}},\\ \prod_{\mathfrak{L}=1}^{\tilde{k}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-O(\mathfrak{L})}}{}^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\mathfrak{E}^{i2\pi\left(\prod_{\mathfrak{L}=1}^{\tilde{k}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-O(\mathfrak{L})}}{}^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)}\end{array}\right)$$

3. Fairly operational laws for CPQROFSs

In this section, we derived the fairly operational laws and their related results.

Definition 6: We discussed some operational laws:

$$\mathbb{Z}_{\mathbb{CQ}-1}\otimes\mathbb{Z}_{\mathbb{CQ}-2}$$
$$=\left(\begin{array}{c}\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}\\ \mathfrak{E}^{i2\pi\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}},\\ \left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)\right)^{\frac{1}{q_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}\\ \mathfrak{E}^{i2\pi\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}\end{array}\right)$$

$$\rho_{\mathbb{CQ}}\mathbb{Z}_{\mathbb{CQ}-1}=\left(\begin{array}{c}\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{q_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}\\ \mathfrak{E}^{i2\pi\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{q_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}},\\ \left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{q_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}\\ \mathfrak{E}^{i2\pi\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{q_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}\end{array}\right)$$

Proposition 1: Based on the $\mathbb{Z}_{\mathbb{CQ}-1}=\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}\right)},\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}\right)}\right)$ and $\mathbb{Z}_{\mathbb{CQ}-2}=\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}\right)},\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}\right)}\right)$, if $\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}=\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}},\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}=\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}$ and $\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}=\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}},\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}=\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}$, thus

$$\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathbb{Z}_{\mathbb{CQ}-1}\otimes\mathbb{Z}_{\mathbb{CQ}-2}}}=\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathbb{Z}_{\mathbb{CQ}-1}\otimes\mathbb{Z}_{\mathbb{CQ}-2}}},\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathbb{Z}_{\mathbb{CQ}-1}\otimes\mathbb{Z}_{\mathbb{CQ}-2}}}=\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathbb{Z}_{\mathbb{CQ}-1}\otimes\mathbb{Z}_{\mathbb{CQ}-2}}}.$$
$$\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\rho_{\mathbb{CQ}}\mathbb{Z}_{\mathbb{CQ}-1}}}=\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\rho_{\mathbb{CQ}}\mathbb{Z}_{\mathbb{CQ}-1}}},\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\rho_{\mathbb{CQ}}\mathbb{Z}_{\mathbb{CQ}-1}}}=\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\rho_{\mathbb{CQ}}\mathbb{Z}_{\mathbb{CQ}-1}}}.$$

Proof: Assume that if $\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}=\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}},\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}=\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}$ and $\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}=\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}},\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}=\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}$, thus

$$\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathbb{Z}_{\mathbb{CQ}-1}\otimes\mathbb{Z}_{\mathbb{CQ}-2}}}}{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathbb{Z}_{\mathbb{CQ}-1}\otimes\mathbb{Z}_{\mathbb{CQ}-2}}}}$$

$$=\frac{\left(\frac{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-2}}}{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-2}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-2}}}\left(1-\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}\right)\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-2}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-2}}\right)\right)\right)^{\frac{1}{q_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}{\left(\frac{\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-2}}}{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-2}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-2}}}\left(1-\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}\right)\left(1-\mathbb{S}^{\mathfrak{Q}}_{\mathbb{Z}_{\mathbb{RP}-2}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-2}}\right)\right)\right)^{\frac{1}{q_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}$$

$$=\left(\frac{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-2}}}{\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-2}}}\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}}}=1$$

Further

$$\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathbb{Z}_{\mathbb{CQ}-1}\otimes\mathbb{Z}_{\mathbb{CQ}-2}}}}{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathbb{Z}_{\mathbb{CQ}-1}\otimes\mathbb{Z}_{\mathbb{CQ}-2}}}}=\frac{\left(\frac{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-2}}}{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-2}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-2}}}\left(1-\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}\right)\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-2}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-2}}\right)\right)\right)^{\frac{1}{q_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}{\left(\frac{\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-2}}}{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-2}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-2}}}\left(1-\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}\right)\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-2}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-2}}\right)\right)\right)^{\frac{1}{q_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}=$$

$$\left(\frac{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-2}}}{\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-2}}}\right)^{\frac{1}{q_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}=1.$$

$$\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\rho_{\mathbb{CQ}}\mathbb{Z}_{\mathbb{CQ}-1}}}}{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\rho_{\mathbb{CQ}}\mathbb{Z}_{\mathbb{CQ}-1}}}}=\frac{\left(\frac{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}}{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}}\left(1-\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{q_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}{\left(\frac{\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}}{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}}\left(1-\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}=\left(\frac{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}}{\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-1}}}\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}=1$$

Further

$$\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\rho_{\mathbb{CQ}}\mathbb{Z}_{\mathbb{CQ}-1}}}}{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\rho_{\mathbb{CQ}}\mathbb{Z}_{\mathbb{CQ}-1}}}}=\frac{\left(\frac{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}}{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}}\left(1-\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}{\left(\frac{\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}}{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}}\left(1-\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}=\left(\frac{\mathbb{S}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}}{\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-1}}}\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}=1.$$

Theorem 1: Some operational laws are stated below:

i. $\mathbb{Z}_{\mathbb{CQ}-1}\otimes\mathbb{Z}_{\mathbb{CQ}-2}=\mathbb{Z}_{\mathbb{CQ}-2}\otimes\mathbb{Z}_{\mathbb{CQ}-1}$;

ii. $\rho_{\mathbb{CQ}}\left(\mathbb{Z}_{\mathbb{CQ}-1}\otimes\mathbb{Z}_{\mathbb{CQ}-2}\right)=\rho_{\mathbb{CQ}}\mathbb{Z}_{\mathbb{CQ}-1}\otimes\rho_{\mathbb{CQ}}\mathbb{Z}_{\mathbb{CQ}-2}$;

iii. $\left(\rho_{\mathbb{CQ}-1}+\rho_{\mathbb{CQ}-2}\right)\mathbb{Z}_{\mathbb{CQ}-1}=\rho_{\mathbb{CQ}-1}\mathbb{Z}_{\mathbb{CQ}-1}\otimes\rho_{\mathbb{CQ}-1}\mathbb{Z}_{\mathbb{CQ}-1}$.

Proof: Let,

1) $\mathbb{Z}_{\mathbb{CQ}-1} \otimes \mathbb{Z}_{\mathbb{CQ}-2} =$

$$\left(\begin{array}{c}\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\ \mathfrak{E}^{i2\pi\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}, \\ \left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\ \mathfrak{E}^{i2\pi\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}\end{array}\right)=$$

$$\left(\begin{array}{c}\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\ \mathfrak{E}^{i2\pi\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}, \\ \left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\ \mathfrak{E}^{i2\pi\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}\end{array}\right)=$$

$\mathbb{Z}_{\mathbb{CQ}-2} \otimes \mathbb{Z}_{\mathbb{CQ}-1}$.

2) $\rho_{\mathbb{CQ}}\left(\mathbb{Z}_{\mathbb{CQ}-1}\otimes\mathbb{Z}_{\mathbb{CQ}-2}\right)=$

$$
\begin{aligned}
\Bigg(&\left(\frac{\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\right)^{\frac{\rho_{\mathbb{CQ}}}{\mathbb{O}_{\mathbb{CQ}}}\mathbb{O}_{\mathbb{CQ}}}}{\left(\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\right)^{\frac{\rho_{\mathbb{CQ}}}{\mathbb{O}_{\mathbb{CQ}}}\mathbb{O}_{\mathbb{CQ}}}+\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\right)^{\frac{\rho_{\mathbb{CQ}}}{\mathbb{O}_{\mathbb{CQ}}}\mathbb{O}_{\mathbb{CQ}}}\right)}\left(1-\left(1-\left(1-\begin{pmatrix}1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\\-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\end{pmatrix}\begin{pmatrix}1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}\\-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\end{pmatrix}\right)\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}\\
&\mathfrak{E}^{i2\pi\left(\frac{\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\right)^{\frac{\rho_{\mathbb{CQ}}}{\mathbb{O}_{\mathbb{CQ}}}\mathbb{O}_{\mathbb{CQ}}}}{\left(\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\right)^{\frac{\rho_{\mathbb{CQ}}}{\mathbb{O}_{\mathbb{CQ}}}\mathbb{O}_{\mathbb{CQ}}}+\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\right)^{\frac{\rho_{\mathbb{CQ}}}{\mathbb{O}_{\mathbb{CQ}}}\mathbb{O}_{\mathbb{CQ}}}\right)}\left(1-\left(1-\left(1-\begin{pmatrix}1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\\-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\end{pmatrix}\begin{pmatrix}1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}\\-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\end{pmatrix}\right)\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}},\\
&\left(\frac{\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\right)^{\frac{\rho_{\mathbb{CQ}}}{\mathbb{O}_{\mathbb{CQ}}}\mathbb{O}_{\mathbb{CQ}}}}{\left(\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\right)^{\frac{\rho_{\mathbb{CQ}}}{\mathbb{O}_{\mathbb{CQ}}}\mathbb{O}_{\mathbb{CQ}}}+\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\right)^{\frac{\rho_{\mathbb{CQ}}}{\mathbb{O}_{\mathbb{CQ}}}\mathbb{O}_{\mathbb{CQ}}}\right)}\left(1-\left(1-\left(1-\begin{pmatrix}1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\\-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\end{pmatrix}\begin{pmatrix}1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}\\-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\end{pmatrix}\right)\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}\\
&\mathfrak{E}^{i2\pi\left(\frac{\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\right)^{\frac{\rho_{\mathbb{CQ}}}{\mathbb{O}_{\mathbb{CQ}}}\mathbb{O}_{\mathbb{CQ}}}}{\left(\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}+\mathbb{T}_{\mathfrak{Q}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\right)^{\frac{\rho_{\mathbb{CQ}}}{\mathbb{O}_{\mathbb{CQ}}}\mathbb{O}_{\mathbb{CQ}}}+\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}}\right)^{\frac{\rho_{\mathbb{CQ}}}{\mathbb{O}_{\mathbb{CQ}}}\mathbb{O}_{\mathbb{CQ}}}\right)}\left(1-\left(1-\left(1-\begin{pmatrix}1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}\\-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\end{pmatrix}\begin{pmatrix}1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}\\-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\end{pmatrix}\right)\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}\Bigg)
\end{aligned}
$$

$$
=\left(\begin{array}{c}
\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\
\mathfrak{E}^{i2\pi\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}, \\
\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\
\mathfrak{E}^{i2\pi\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}
\end{array}\right)
$$

Additionally,

$\rho_{\mathbb{CQ}}\mathbb{Z}_{\mathbb{CQ}-1} \otimes \rho_{\mathbb{CQ}}\mathbb{Z}_{\mathbb{CQ}-2}$

$$
=\left(\begin{array}{c}
\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\
\mathfrak{E}^{i2\pi\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}, \\
\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\
\mathfrak{E}^{i2\pi\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}
\end{array}\right)
\otimes\left(\begin{array}{c}
\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}c}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\
\mathfrak{E}^{i2\pi\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}, \\
\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\
\mathfrak{E}^{i2\pi\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}\rho_{\mathbb{CQ}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}
\end{array}\right)
$$

$$= \left(\begin{array}{c} \left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}} \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}} \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}} + \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}} \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}} \left(1 - \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}} \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\ \mathfrak{E}^{i2\pi \left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}} \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}} \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}} + \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}} \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}} \left(1 - \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}} \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}} , \\ \left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}} \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}} \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}} + \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}} \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}} \left(1 - \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}} \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\ \mathfrak{E}^{i2\pi \left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}} \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}} \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\rho_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}} + \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}} \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\rho_{\mathbb{CQ}}\mathbb{O}_{\mathbb{CQ}}}} \left(1 - \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}} \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-2}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\rho_{\mathbb{CQ}}}\right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}} \end{array} \right)$$

3) Omitted.

4. CPQROF weighted fairly aggregation operators

In this section we evaluated the CPQROFWFA and CPQROFOWFA operators, and discussed their basic properties:

Definition 7: A CPQROFWFA operator is shaped in the form:

$$CPQROFWFA\left(\mathbb{Z}_{\mathbb{CQ}-1}, \mathbb{Z}_{\mathbb{CQ}-2}, \dots, \mathbb{Z}_{\mathbb{CQ}-\tilde{k}}\right) = \left(w_{\mathbb{Z}_{Wv-1}} \mathbb{Z}_{\mathbb{CQ}-1}\right) \otimes \left(w_{\mathbb{Z}_{Wv-2}} \mathbb{Z}_{\mathbb{CQ}-2}\right) \otimes \dots \otimes \left(w_{\mathbb{Z}_{Wv-\tilde{k}}} \mathbb{Z}_{\mathbb{CQ}-\tilde{k}}\right) \quad (12)$$

Theorem 2: We proved that the aggregated value of the above information is again a CPQROFN, such as

$CPQROFWFA\left(\mathbb{Z}_{\mathbb{CQ}-1}, \mathbb{Z}_{\mathbb{CQ}-2}, \dots, \mathbb{Z}_{\mathbb{CQ}-\tilde{k}}\right) =$

$$\left(\begin{array}{c} \left(\frac{\prod_{\mathfrak{L}=1}^{\tilde{k}} \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{\tilde{k}} \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} + \prod_{\mathfrak{L}=1}^{\tilde{k}} \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}} \left(1 - \prod_{\mathfrak{L}=1}^{\tilde{k}} \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\ \mathfrak{E}^{i2\pi \left(\frac{\prod_{\mathfrak{L}=1}^{\tilde{k}} \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{\tilde{k}} \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} + \prod_{\mathfrak{L}=1}^{\tilde{k}} \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}} \left(1 - \prod_{\mathfrak{L}=1}^{\tilde{k}} \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}} , \\ \left(\frac{\prod_{\mathfrak{L}=1}^{\tilde{k}} \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{\tilde{k}} \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} + \prod_{\mathfrak{L}=1}^{\tilde{k}} \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}} \left(1 - \prod_{\mathfrak{L}=1}^{\tilde{k}} \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\ \mathfrak{E}^{i2\pi \left(\frac{\prod_{\mathfrak{L}=1}^{\tilde{k}} \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{\tilde{k}} \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} + \prod_{\mathfrak{L}=1}^{\tilde{k}} \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}} \left(1 - \prod_{\mathfrak{L}=1}^{\tilde{k}} \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}} \end{array} \right) \quad (1)$$

Proof:

1) If $\tilde{k} = 1$, thus

$$\mathbb{CQ}ROFWFA\left(\mathbb{Z}_{\mathbb{CQ}-1}, \mathbb{Z}_{\mathbb{CQ}-2}, \dots, \mathbb{Z}_{\mathbb{CQ}-\tilde{k}}\right) = w_{\mathbb{Z}_{Wv-1}} \mathbb{Z}_{\mathbb{CQ}-1}$$

$$= \left(\begin{array}{c} \left(\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}} w_{\mathbb{Z}_{Wv-1}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}} w_{\mathbb{Z}_{Wv-1}}} + \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}} w_{\mathbb{Z}_{Wv-1}}}} \left(1 - \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-1}}} \right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}} \mathfrak{Q}_{\mathbb{CQ}}}} \right. \\ \left. \mathfrak{E}^{i2\pi \left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}} w_{\mathbb{Z}_{Wv-1}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}} w_{\mathbb{Z}_{Wv-1}}} + \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}} w_{\mathbb{Z}_{Wv-1}}}} \left(1 - \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-1}}} \right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}} \mathfrak{Q}_{\mathbb{CQ}}}}} \right), \\ \left(\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}} w_{\mathbb{Z}_{Wv-1}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}} w_{\mathbb{Z}_{Wv-1}}} + \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}} w_{\mathbb{Z}_{Wv-1}}}} \left(1 - \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-1}}^{\mathbb{O}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-1}}} \right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}} \mathfrak{Q}_{\mathbb{CQ}}}} \right. \\ \left. \mathfrak{E}^{i2\pi \left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}} w_{\mathbb{Z}_{Wv-1}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}} w_{\mathbb{Z}_{Wv-1}}} + \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}} w_{\mathbb{Z}_{Wv-1}}}} \left(1 - \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-1}}^{\mathbb{O}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-1}}} \right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}} \mathfrak{Q}_{\mathbb{CQ}}}}} \right) \end{array} \right)$$

For $\tilde{k} = 1$, we hold

2) If $\tilde{k} = k$, thus

$$\mathbb{CQ}ROFWFA(\mathbb{Z}_{\mathbb{CQ}-1}, \mathbb{Z}_{\mathbb{CQ}-2}, \dots, \mathbb{Z}_{\mathbb{CQ}-k})$$

$$= \left(\begin{array}{c} \left(\left(\frac{\prod_{\mathfrak{L}=1}^{k} \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{k} \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} + \prod_{\mathfrak{L}=1}^{k} \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}} \left(1 - \prod_{\mathfrak{L}=1}^{k} \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} \right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}} \mathfrak{Q}_{\mathbb{CQ}}}} \right. \\ \left. \mathfrak{E}^{i2\pi \left(\frac{\prod_{\mathfrak{L}=1}^{k} \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{k} \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} + \prod_{\mathfrak{L}=1}^{k} \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}} \left(1 - \prod_{\mathfrak{L}=1}^{k} \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} \right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}} \mathfrak{Q}_{\mathbb{CQ}}}}} \right), \\ \left(\left(\frac{\prod_{\mathfrak{L}=1}^{k} \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{k} \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} + \prod_{\mathfrak{L}=1}^{k} \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}} \left(1 - \prod_{\mathfrak{L}=1}^{k} \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} \right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}} \mathfrak{Q}_{\mathbb{CQ}}}} \right. \\ \left. \mathfrak{E}^{i2\pi \left(\frac{\prod_{\mathfrak{L}=1}^{\tilde{k}} \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{k} \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} + \prod_{\mathfrak{L}=1}^{k} \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}} \left(1 - \prod_{\mathfrak{L}=1}^{\tilde{k}} \left(1 - \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}} - \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}} \right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} \right) \right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}} \mathfrak{Q}_{\mathbb{CQ}}}}} \right) \end{array} \right)$$

3) If $\tilde{k} = k + 1$, thus

$$CPQROFWFA(\mathbb{Z}_{\mathbb{CQ}-1}, \mathbb{Z}_{\mathbb{CQ}-2}, \dots, \mathbb{Z}_{\mathbb{CQ}-k+1}) = CPQROFWFA(\mathbb{Z}_{\mathbb{CQ}-1}, \mathbb{Z}_{\mathbb{CQ}-2}, \dots, \mathbb{Z}_{\mathbb{CQ}-k}) \otimes w_{\mathbb{Z}_{Wv-k+1}} \mathbb{Z}_{\mathbb{CQ}-k+1}$$

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$$
= \left(\begin{array}{c}
\left(\dfrac{\prod_{\mathfrak{L}=1}^{k}\left(\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}{\prod_{\mathfrak{L}=1}^{k}\left(\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}+\prod_{\mathfrak{L}=1}^{k}\left(\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}\left(1-\prod_{\mathfrak{L}=1}^{k}\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\
\mathfrak{E}^{i2\pi\left(\frac{\prod_{\mathfrak{L}=1}^{k}\left(\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}{\prod_{\mathfrak{L}=1}^{k}\left(\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}+\prod_{\mathfrak{L}=1}^{k}\left(\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}\left(1-\prod_{\mathfrak{L}=1}^{k}\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}, \\
\left(\dfrac{\prod_{\mathfrak{L}=1}^{k}\left(\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}{\prod_{\mathfrak{L}=1}^{k}\left(\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}+\prod_{\mathfrak{L}=1}^{k}\left(\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}\left(1-\prod_{\mathfrak{L}=1}^{k}\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\
\mathfrak{E}^{i2\pi\left(\frac{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}{\prod_{\mathfrak{L}=1}^{k}\left(\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}+\prod_{\mathfrak{L}=1}^{k}\left(\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}\left(1-\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}
\end{array}\right)
$$

$$
\otimes \left(\begin{array}{c}
\left(\dfrac{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}w\mathbb{Z}_{Wv-k+1}}_{\mathbb{Z}_{\mathbb{RP}-k+1}}}{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}w\mathbb{Z}_{Wv-k+1}}_{\mathbb{Z}_{\mathbb{RP}-k+1}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}w\mathbb{Z}_{Wv-k+1}}_{\mathbb{Z}_{\mathbb{RP}-k+1}}}\left(1-\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-k+1}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-k+1}}\right)^{w\mathbb{Z}_{Wv-k+1}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\
\mathfrak{E}^{i2\pi\left(\frac{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}w\mathbb{Z}_{Wv-k+1}}_{\mathbb{Z}_{\mathbb{IP}-k+1}}}{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}w\mathbb{Z}_{Wv-k+1}}_{\mathbb{Z}_{\mathbb{IP}-k+1}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}w\mathbb{Z}_{Wv-k+1}}_{\mathbb{Z}_{\mathbb{IP}-k+1}}}\left(1-\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-k+1}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-k+1}}\right)^{w\mathbb{Z}_{Wv-k+1}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}, \\
\left(\dfrac{\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}w\mathbb{Z}_{Wv-k+1}}_{\mathbb{Z}_{\mathbb{RP}-k+1}}}{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}w\mathbb{Z}_{Wv-k+1}}_{\mathbb{Z}_{\mathbb{RP}-k+1}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}w\mathbb{Z}_{Wv-k+1}}_{\mathbb{Z}_{\mathbb{RP}-k+1}}}\left(1-\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-k+1}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-k+1}}\right)^{w\mathbb{Z}_{Wv-k+1}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\
\mathfrak{E}^{i2\pi\left(\frac{\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}w\mathbb{Z}_{Wv-k+1}}_{\mathbb{Z}_{\mathbb{IP}-k+1}}}{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}w\mathbb{Z}_{Wv-k+1}}_{\mathbb{Z}_{\mathbb{IP}-k+1}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}w\mathbb{Z}_{Wv-k+1}}_{\mathbb{Z}_{\mathbb{IP}-k+1}}}\left(1-\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-k+1}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-k+1}}\right)^{w\mathbb{Z}_{Wv-k+1}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}
\end{array}\right)
$$

$$
= \left(\begin{array}{c}
\left(\dfrac{\prod_{\mathfrak{L}=1}^{k+1}\left(\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}{\prod_{\mathfrak{L}=1}^{k+1}\left(\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}+\prod_{\mathfrak{L}=1}^{k+1}\left(\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}\left(1-\prod_{\mathfrak{L}=1}^{k+1}\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\
\mathfrak{E}^{i2\pi\left(\frac{\prod_{\mathfrak{L}=1}^{k+1}\left(\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}{\prod_{\mathfrak{L}=1}^{k+1}\left(\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}+\prod_{\mathfrak{L}=1}^{k+1}\left(\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}\left(1-\prod_{\mathfrak{L}=1}^{k+1}\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}, \\
\left(\dfrac{\prod_{\mathfrak{L}=1}^{k+1}\left(\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}{\prod_{\mathfrak{L}=1}^{k+1}\left(\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}+\prod_{\mathfrak{L}=1}^{k+1}\left(\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}\left(1-\prod_{\mathfrak{L}=1}^{k+1}\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\
\mathfrak{E}^{i2\pi\left(\frac{\prod_{\mathfrak{L}=1}^{k+1}\left(\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}{\prod_{\mathfrak{L}=1}^{k+1}\left(\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}+\prod_{\mathfrak{L}=1}^{k+1}\left(\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}}\left(1-\prod_{\mathfrak{L}=1}^{k+1}\left(1-\mathbb{S}^{\mathfrak{Q}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)^{w\mathbb{Z}_{Wv-\mathfrak{L}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}
\end{array}\right)
$$

we proved our required results $\tilde{k}$.

Property 1: When $\mathbb{Z}_{\mathbb{CQ}-\mathfrak{L}} = \mathbb{Z}_{\mathbb{CQ}}\ that\ is,\ \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}} = \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}, \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}} = \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}, \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}} = \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}$, and $\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}} = \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}$, thus

$$CPQROFWFA\left(\mathbb{Z}_{\mathbb{CQ}-1}, \mathbb{Z}_{\mathbb{CQ}-2}, \ldots, \mathbb{Z}_{\mathbb{CQ}-\tilde{k}}\right) = \mathbb{Z}_{\mathbb{CQ}} \quad (14)$$

Proof: Let $\mathbb{Z}_{\mathbb{CQ}-\mathfrak{L}} = \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)}, \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)}\right), \mathfrak{L} = 1,2,..,\tilde{k}$, when $\mathbb{Z}_{\mathbb{CQ}-\mathfrak{L}} = \mathbb{Z}_{\mathbb{CQ}} = \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}\right)}, \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}\right)}\right)$, i.e., $\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}} = \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}, \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}} = \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}, \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}} = \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}$, and $\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}} = \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}$, thus

$$CPQROFWFA\left(\mathbb{Z}_{\mathbb{CQ}-1}, \mathbb{Z}_{\mathbb{CQ}-2}, \ldots, \mathbb{Z}_{\mathbb{CQ}-\tilde{k}}\right)$$

$$= \left(\begin{array}{c} \left(\frac{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} + \prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}\left(1-\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\ \mathfrak{E}^{i2\pi\left(\frac{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}+\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}\left(1-\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}, \\ \left(\frac{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} + \prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}\left(1-\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\ \mathfrak{E}^{i2\pi\left(\frac{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}+\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}\left(1-\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}} \end{array}\right)$$

$$= \left(\begin{array}{c} \left(\frac{\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} + \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\ \mathfrak{E}^{i2\pi\left(\frac{\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}+\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}, \\ \left(\frac{\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}} + \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\ \mathfrak{E}^{i2\pi\left(\frac{\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}+\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}\left(1-\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}} \end{array}\right)$$

$$= \left(\begin{array}{c} \left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-1+\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \mathfrak{E}^{i2\pi\left(\frac{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-1+\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}, \\ \left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-1+\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \mathfrak{E}^{i2\pi\left(\frac{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}}{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}}\left(1-1+\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}} \end{array} \right) = \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}\right)}, \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}\right)}\right) = \mathbb{Z}_{\mathbb{CQ}}$$

Property 2: Some properties are stated below:

i. $\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\} \leq \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}} \leq \max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}, \min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\} \leq \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}} \leq \max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}$.

iv. $\frac{\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{q_{\mathbb{CQ}}}\right\}\times\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}{\max_i\left\{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}+\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}} \leq \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}} \leq \min_i\left\{\frac{\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}\times\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}{\min_i\left\{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}+\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}, 1\right\}, \frac{\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}\times\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}{\max_i\left\{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}+\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}} \leq \mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}} \leq \min_i\left\{\frac{\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}\times\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}{\min_i\left\{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}+\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}, 1\right\}$.

v. $\frac{\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}\times\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}{\max_i\left\{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}+\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}} \leq \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}} \leq \min_i\left\{\frac{\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}\times\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}{\min_i\left\{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}+\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}, 1\right\}, \frac{\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}\times\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}{\max_i\left\{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}+\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}} \leq \mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}} \leq \min_i\left\{\frac{\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}\times\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}{\min_i\left\{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}+\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}, 1\right\}$.

Property 3: when $\mathbb{Z}_{\mathbb{CQ}-\mathfrak{L}} \leq \mathbb{Z}'_{\mathbb{CQ}-\mathfrak{L}}$, thus

$$CPQROFWFA\left(\mathbb{Z}_{\mathbb{CQ}-1}, \mathbb{Z}_{\mathbb{CQ}-2}, \dots, \mathbb{Z}_{\mathbb{CQ}-\tilde{k}}\right) \leq CPQROFWFA\left(\mathbb{Z}'_{\mathbb{CQ}-1}, \mathbb{Z}'_{\mathbb{CQ}-2}, \dots, \mathbb{Z}'_{\mathbb{CQ}-\tilde{k}}\right) \tag{2}$$

Definition 8: For any family of CQROFNs $\mathbb{Z}_{\mathbb{CQ}-\mathfrak{L}} = \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)}, \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)}\right), \mathfrak{L} = 1,2,..,\tilde{k}$, then the CQROFOWFA operator is demonstrated by:

$$\mathbb{CQ}ROFOWFA\left(\mathbb{Z}_{\mathbb{CQ}-1}, \mathbb{Z}_{\mathbb{CQ}-2}, \dots, \mathbb{Z}_{\mathbb{CQ}-\tilde{k}}\right) = \left(w_{\mathbb{Z}_{Wv-1}}\mathbb{Z}_{\mathbb{CQ}-\sigma(1)}\right) \otimes \left(w_{\mathbb{Z}_{Wv-2}}\mathbb{Z}_{\mathbb{CQ}-\sigma(2)}\right) \otimes \dots \otimes \left(w_{\mathbb{Z}_{Wv-\tilde{k}}}\mathbb{Z}_{\mathbb{CQ}-\sigma(\tilde{k})}\right) \tag{3}$$

Where $w_{\mathbb{Z}_{Wv}} = \left\{w_{\mathbb{Z}_{Wv-1}}, w_{\mathbb{Z}_{Wv-2}}, \dots, w_{\mathbb{Z}_{Wv-\tilde{k}}}\right\}$ with a rule that is $\sum_{\mathfrak{L}=1}^{\tilde{k}} w_{\mathbb{Z}_{Wv-\mathfrak{L}}} = 1, w_{\mathbb{Z}_{Wv-\mathfrak{L}}} \in [0,1]$.

Theorem 3: Based on the $\mathbb{Z}_{\mathbb{CQ}-\mathfrak{L}} = \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)}, \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)}\right), \mathfrak{L} = 1,2,..,\tilde{k}$, then by using the Eq. (3), we get

$$\mathbb{CQ}ROFOWFA\left(\mathbb{Z}_{\mathbb{CQ}-1},\mathbb{Z}_{\mathbb{CQ}-2},\dots,\mathbb{Z}_{\mathbb{CQ}-\tilde{k}}\right)=$$

$$\left(\begin{array}{c}\left(\frac{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}+\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}\left(1-\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\ \mathfrak{E}^{i2\pi\left(\frac{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}+\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}\left(1-\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}, \\ \left(\frac{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}+\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}\left(1-\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \\ \mathfrak{E}^{i2\pi\left(\frac{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}{\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}+\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}}\left(1-\prod_{\mathfrak{L}=1}^{\tilde{k}}\left(1-\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}-\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right)^{w_{\mathbb{Z}_{Wv-\mathfrak{L}}}}\right)\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}\end{array}\right) \tag{4}$$

Where $w_{\mathbb{Z}_{Wv}}=\left\{w_{\mathbb{Z}_{Wv-1}},w_{\mathbb{Z}_{Wv-2}},\dots,w_{\mathbb{Z}_{Wv-\tilde{k}}}\right\}$ with a rule that is $\sum_{\mathfrak{L}=1}^{\tilde{k}}w_{\mathbb{Z}_{Wv-\mathfrak{L}}}=1, w_{\mathbb{Z}_{Wv-\mathfrak{L}}}\in[0,1]$ with the permutation $\sigma(\mathfrak{L})$ of $\left(\mathfrak{L}=1,2,..,\tilde{k}\right)$ with a rule that is $\sigma(\mathfrak{L}-1)\geq\sigma(\mathfrak{L})$

Property 4: Based on any two families of CQROFNs $\mathbb{Z}_{\mathbb{CQ}-\mathfrak{L}}=\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)},\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)}\right),\mathfrak{L}=1,2,..,\tilde{k},$ if $\mathbb{Z}_{\mathbb{CQ}-\mathfrak{L}}=\mathbb{Z}_{\mathbb{CQ}}=\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}\right)},\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}\right)}\right)$, i.e., $\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}=\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}},\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}=\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}},\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}=\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}$, and $\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}=\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}$, then

$$\mathbb{CQ}ROFOWFA\left(\mathbb{Z}_{\mathbb{CQ}-1},\mathbb{Z}_{\mathbb{CQ}-2},\dots,\mathbb{Z}_{\mathbb{CQ}-\tilde{k}}\right)=\mathbb{Z}_{\mathbb{CQ}} \tag{5}$$

Property 5: Based on any two families of CQROFNs $\mathbb{Z}_{\mathbb{CQ}-\mathfrak{L}}=\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)},\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)}\right),\mathfrak{L}=1,2,..,\tilde{k},$

and $\mathbb{CQ}ROFOWFA\left(\mathbb{Z}_{\mathbb{CQ}-1},\mathbb{Z}_{\mathbb{CQ}-2},\dots,\mathbb{Z}_{\mathbb{CQ}-\tilde{k}}\right)=\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}\right)},\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}\right)}\right)$, then

i. $\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}\leq\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}\leq\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right\},\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}\leq\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathbb{O}_{\mathbb{CQ}}}\leq\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}$.

ii. $\frac{\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}\times\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}{\max_i\left\{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}+\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}\leq\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}\leq\min_i\left\{\frac{\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}\times\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}{\min_i\left\{\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}+\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}},1\right\},\frac{\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}\times\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}{\max_i\left\{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}+\min_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}\leq\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}\leq\min_i\left\{\frac{\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}+\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}\times\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}}{\min_i\left\{\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathbb{O}_{\mathbb{CQ}}}\right\}+\max_i\left\{\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}^{\mathfrak{Q}_{\mathbb{CQ}}}\right\}},1\right\}$.

iii. $\frac{\min_i\left\{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}\right\}\times\min_i\left\{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}\right\}}{\max_i\left\{\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}\right\}+\min_i\left\{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}\right\}} \leq \mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}}} \leq$
$\min_i\left\{\frac{\max_i\left\{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}\right\}\times\max_i\left\{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}\right\}}{\min_i\left\{\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}\right\}+\max_i\left\{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}-\sigma(\mathfrak{L})}}\right\}},1\right\}, \frac{\min_i\left\{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}\right\}\times\min_i\left\{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}\right\}}{\max_i\left\{\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}\right\}+\min_i\left\{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}\right\}} \leq$
$\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}}} \leq \min_i\left\{\frac{\max_i\left\{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}+\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}\right\}\times\max_i\left\{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}\right\}}{\min_i\left\{\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}\right\}+\max_i\left\{\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}-\sigma(\mathfrak{L})}}\right\}},1\right\}.$

Property 6: Based on any two families of CQROFNs $\mathbb{Z}'_{\mathbb{CQ}-\mathfrak{L}} = \left(\mathbb{S}'_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}'_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)}, \mathbb{T}'_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}'_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)}\right)$ and
$\mathbb{Z}_{\mathbb{CQ}-\mathfrak{L}} = \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)}, \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}-\mathfrak{L}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}-\mathfrak{L}}}\right)}\right), \mathfrak{L} = 1,2,..,\tilde{k}$, if $\mathbb{Z}_{\mathbb{CQ}-\mathfrak{L}} \leq \mathbb{Z}'_{\mathbb{CQ}-\mathfrak{L}}$, then

$$\mathbb{CQ}ROFOWFA\left(\mathbb{Z}_{\mathbb{CQ}-1}, \mathbb{Z}_{\mathbb{CQ}-2}, \dots, \mathbb{Z}_{\mathbb{CQ}-\tilde{k}}\right) \leq \mathbb{CQ}ROFOWFA\left(\mathbb{Z}'_{\mathbb{CQ}-1}, \mathbb{Z}'_{\mathbb{CQ}-2}, \dots, \mathbb{Z}'_{\mathbb{CQ}-\tilde{k}}\right)$$

5. Application proposed theory in decision-making problems

In this section, we calculate the system of the MADM problem based on the initiated technique, called the CPQROFWFA operator to find the supremacy and validity of the proposed theory.

For this, we consider the family of alternatives $\mathbb{Z}_{\mathbb{CQ}-1}, \mathbb{Z}_{\mathbb{CQ}-2}, \dots, \mathbb{Z}_{\mathbb{CQ}-\tilde{k}}$ and for each alternative, we have considered the collection of attributes, such as $\mathbb{Z}_{AT-1}, \mathbb{Z}_{AT-2}, \dots, \mathbb{Z}_{AT-\tilde{m}}$ with weight vectors, such as $\mathcal{w}_{\mathbb{Z}_{Wv}} = \left\{\mathcal{w}_{\mathbb{Z}_{Wv-1}}, \mathcal{w}_{\mathbb{Z}_{Wv-2}}, \dots, \mathcal{w}_{\mathbb{Z}_{Wv-\tilde{k}}}\right\}$ with a rule that is $\sum_{\mathfrak{L}=1}^{\tilde{k}} \mathcal{w}_{\mathbb{Z}_{Wv-\mathfrak{L}}} = 1, \mathcal{w}_{\mathbb{Z}_{Wv-\mathfrak{L}}} \in [0,1]$. Further, we will compute the matrix with the help of the collection of the CPQROFNs, such. $\mathbb{S}_{\mathbb{Z}_{\mathbb{CQ}}}(\tilde{x}) = \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}(\tilde{x})\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}(\tilde{x})\right)}$ and $\mathbb{T}_{\mathbb{Z}_{\mathbb{CQ}}}(\tilde{x}) = \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}(\tilde{x})\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}(\tilde{x})\right)}$ represents the supporting and supporting against grade with $0 \leq \mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}}}(\tilde{x}) + \mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}}}(\tilde{x}) \leq 1$ and $0 \leq \mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}}}(\tilde{x}) + \mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}}}(\tilde{x}) \leq 1, \mathbb{O}_{\mathbb{CQ}}, \mathfrak{Q}_{\mathbb{CQ}} \geq 1$. further, the terms $\mathcal{J}_{\mathbb{Z}_{\mathbb{CQ}}}(\tilde{x}) = \mathcal{J}_{\mathbb{Z}_{\mathbb{RP}}}(\tilde{x})\mathfrak{E}^{i2\pi\left(\mathcal{J}_{\mathbb{Z}_{\mathbb{IP}}}\right)} = \left(1 - \mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}}}(\tilde{x}) - \mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}}}(\tilde{x})\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}} \mathfrak{E}^{i2\pi\left(1-\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}}}(\tilde{x})-\mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}}}(\tilde{x})\right)^{\frac{1}{\mathbb{O}_{\mathbb{CQ}}\mathfrak{Q}_{\mathbb{CQ}}}}}$ shows the indeterminacy. Finally, the technique: $\mathbb{Z}_{\mathbb{CQ}} = \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}\right)}, \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}\right)}\right)$, called CPQROFS. Then the procedure of the decision-making technique is listed below:

Step 1: Compute the decision-matrix with the help of the collection of the CPQROFNs, such as

$$X = \left[x_{ij}\right]_{m\times n}$$

Further, we normalized the matrix if needed with the help of below theory, such as

$$N = \begin{cases} \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}\right)}, \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}\right)}\right) & for\ b\mathfrak{E}n\mathfrak{E}fit \\ \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}\right)}, \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}\right)}\right) & for\ cost \end{cases}$$

But in the case of benefit type of data, we do not do it.

Step 2: Aggregate the information in the decision matrix based on the proposed CPQROFWFA operator into a singleton set.

Step 3: Find the score values of the aggregated values with the below theory, such as

$$\mathfrak{S}_{\mathbb{CQ}}\left(\mathbb{Z}_{\mathbb{CQ}}\right) = \frac{1}{2}\left(\mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}}} + \mathbb{S}^{\mathfrak{Q}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}}} - \mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{RP}}} - \mathbb{T}^{\mathbb{O}_{\mathbb{CQ}}}_{\mathbb{Z}_{\mathbb{IP}}}\right), \mathfrak{S}_{\mathbb{CQ}}\left(\mathbb{Z}_{\mathbb{CQ}}\right) \in [-1,1]$$

Step 4: Evaluate the ranking values of the above alternatives according to their score values for deriving the best one.

Further, we simplify the above procedure with the help of some suitable examples. For this, we select the below theory for evaluating the supremacy and validity of the initiated theory.

5.1 Illustrative example

A company wants to invest their money with some valuable and good company, for this, we visit different companies for investment. All these companies contain different features and based on their features, we will select which one is good and which one is bad for investment, the details of the companies are stated below:

iv. Car Company.
v. Mobile Company.
vi. Computer company.
vii. Invest in agriculture.
viii. Property business.

For this, we have also some criteria, such as growth analysis, social impact, political impact, environmental impact, and taxies ratio of the government. Then the procedure of the decision-making technique is listed below:

Step 1: Compute the decision matrix with the help of the collection of the CPQROFNs, see Table 1.

Table 1
CPQROF decision matrix

	$\mathbb{Z}_{AT-1}$	$\mathbb{Z}_{AT-2}$	$\mathbb{Z}_{AT-3}$	$\mathbb{Z}_{AT-4}$	$\mathbb{Z}_{AT-5}$
$\mathbb{Z}_{\mathbb{CQ}-1}$	$\begin{pmatrix}0.5\mathfrak{E}^{i2\pi(0.6)},\\0.5\mathfrak{E}^{i2\pi(0.4)}\end{pmatrix}$	$\begin{pmatrix}0.51\mathfrak{E}^{i2\pi(0.61)},\\0.51\mathfrak{E}^{i2\pi(0.41)}\end{pmatrix}$	$\begin{pmatrix}0.52\mathfrak{E}^{i2\pi(0.62)},\\0.52\mathfrak{E}^{i2\pi(0.42)}\end{pmatrix}$	$\begin{pmatrix}0.53\mathfrak{E}^{i2\pi(0.63)},\\0.53\mathfrak{E}^{i2\pi(0.43)}\end{pmatrix}$	$\begin{pmatrix}0.54\mathfrak{E}^{i2\pi(0.64)},\\0.54\mathfrak{E}^{i2\pi(0.44)}\end{pmatrix}$
$\mathbb{Z}_{\mathbb{CQ}-2}$	$\begin{pmatrix}0.4\mathfrak{E}^{i2\pi(0.5)},\\0.3\mathfrak{E}^{i2\pi(0.2)}\end{pmatrix}$	$\begin{pmatrix}0.41\mathfrak{E}^{i2\pi(0.51)},\\0.31\mathfrak{E}^{i2\pi(0.21)}\end{pmatrix}$	$\begin{pmatrix}0.42\mathfrak{E}^{i2\pi(0.52)},\\0.32\mathfrak{E}^{i2\pi(0.22)}\end{pmatrix}$	$\begin{pmatrix}0.43\mathfrak{E}^{i2\pi(0.53)},\\0.33\mathfrak{E}^{i2\pi(0.23)}\end{pmatrix}$	$\begin{pmatrix}0.44\mathfrak{E}^{i2\pi(0.54)},\\0.34\mathfrak{E}^{i2\pi(0.24)}\end{pmatrix}$
$\mathbb{Z}_{\mathbb{CQ}-3}$	$\begin{pmatrix}0.7\mathfrak{E}^{i2\pi(0.8)},\\0.2\mathfrak{E}^{i2\pi(0.3)}\end{pmatrix}$	$\begin{pmatrix}0.71\mathfrak{E}^{i2\pi(0.81)},\\0.21\mathfrak{E}^{i2\pi(0.31)}\end{pmatrix}$	$\begin{pmatrix}0.72\mathfrak{E}^{i2\pi(0.82)},\\0.22\mathfrak{E}^{i2\pi(0.32)}\end{pmatrix}$	$\begin{pmatrix}0.73\mathfrak{E}^{i2\pi(0.83)},\\0.23\mathfrak{E}^{i2\pi(0.33)}\end{pmatrix}$	$\begin{pmatrix}0.74\mathfrak{E}^{i2\pi(0.84)},\\0.24\mathfrak{E}^{i2\pi(0.34)}\end{pmatrix}$
$\mathbb{Z}_{\mathbb{CQ}-4}$	$\begin{pmatrix}0.8\mathfrak{E}^{i2\pi(0.8)},\\0.2\mathfrak{E}^{i2\pi(0.3)}\end{pmatrix}$	$\begin{pmatrix}0.71\mathfrak{E}^{i2\pi(0.81)},\\0.21\mathfrak{E}^{i2\pi(0.31)}\end{pmatrix}$	$\begin{pmatrix}0.72\mathfrak{E}^{i2\pi(0.82)},\\0.22\mathfrak{E}^{i2\pi(0.32)}\end{pmatrix}$	$\begin{pmatrix}0.73\mathfrak{E}^{i2\pi(0.83)},\\0.23\mathfrak{E}^{i2\pi(0.33)}\end{pmatrix}$	$\begin{pmatrix}0.74\mathfrak{E}^{i2\pi(0.84)},\\0.24\mathfrak{E}^{i2\pi(0.34)}\end{pmatrix}$
$\mathbb{Z}_{\mathbb{CQ}-5}$	$\begin{pmatrix}0.7\mathfrak{E}^{i2\pi(0.6)},\\0.5\mathfrak{E}^{i2\pi(0.4)}\end{pmatrix}$	$\begin{pmatrix}0.71\mathfrak{E}^{i2\pi(0.81)},\\0.21\mathfrak{E}^{i2\pi(0.31)}\end{pmatrix}$	$\begin{pmatrix}0.72\mathfrak{E}^{i2\pi(0.82)},\\0.22\mathfrak{E}^{i2\pi(0.32)}\end{pmatrix}$	$\begin{pmatrix}0.73\mathfrak{E}^{i2\pi(0.83)},\\0.23\mathfrak{E}^{i2\pi(0.33)}\end{pmatrix}$	$\begin{pmatrix}0.74\mathfrak{E}^{i2\pi(0.84)},\\0.24\mathfrak{E}^{i2\pi(0.34)}\end{pmatrix}$

$$X = [x_{ij}]_{m \times n}$$

Further, we normalized the matrix if needed with the help of below theory, such as

$$N = \begin{cases} \left(\mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}\right)}, \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}\right)}\right) & for\ benefit \\ \left(\mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{T}_{\mathbb{Z}_{\mathbb{IP}}}\right)}, \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}\mathfrak{E}^{i2\pi\left(\mathbb{S}_{\mathbb{Z}_{\mathbb{IP}}}\right)}\right) & for\ cost \end{cases}$$

But in the case of benefit types of data, we do not do it. However, the data in Table 1 does not need to be normalized.

Step 2: Aggregate the information in the decision matrix based on the proposed CPQROFWFA operator into a singleton set, see Table 2.

Table 2
CPQROF aggregated matrix

	$\mathbb{Z}_{\mathbb{CQ}-\mathbf{1}}$	$\mathbb{Z}_{\mathbb{CQ}-\mathbf{2}}$	$\mathbb{Z}_{\mathbb{CQ}-\mathbf{3}}$	$\mathbb{Z}_{\mathbb{CQ}-\mathbf{4}}$	$\mathbb{Z}_{\mathbb{CQ}-\mathbf{5}}$
CPQROFWFA operator	$\begin{pmatrix} 0.514592983 \\ \mathfrak{E}^{i2\pi(0.644116809)}, \\ 0.51459298 \\ \mathfrak{E}^{i2\pi(0.09174568)} \end{pmatrix}$	$\begin{pmatrix} 0.445 \\ \mathfrak{E}^{i2\pi(0.51843)}, \\ 0.11402 \\ \mathfrak{E}^{i2\pi(0.00697)} \end{pmatrix}$	$\begin{pmatrix} 0.71599 \\ \mathfrak{E}^{i2\pi(0.81919)}, \\ 0.00189 \\ \mathfrak{E}^{i2\pi(0.00738)} \end{pmatrix}$	$\begin{pmatrix} 0.81564 \\ \mathfrak{E}^{i2\pi(0.81919)}, \\ 0.00112 \\ \mathfrak{E}^{i2\pi(0.00738)} \end{pmatrix}$	$\begin{pmatrix} 0.75821 \\ \mathfrak{E}^{i2\pi(0.64412)}, \\ 0.14885 \\ \mathfrak{E}^{i2\pi(0.09175)} \end{pmatrix}$

Step 3: Find the score values of the aggregated values with the theory below, see Table 3.

Table 3
Representation of score values

	$\mathbb{Z}_{\mathbb{CQ}-\mathbf{1}}$	$\mathbb{Z}_{\mathbb{CQ}-\mathbf{2}}$	$\mathbb{Z}_{\mathbb{CQ}-\mathbf{3}}$	$\mathbb{Z}_{\mathbb{CQ}-\mathbf{4}}$	$\mathbb{Z}_{\mathbb{CQ}-\mathbf{5}}$
CPQROFWFA operator	0.086029964	0.05564	0.35657	0.44646	0.25103

Step 4: Evaluate the ranking values of the above alternatives according to their score values for deriving the best one, such as

$$\mathbb{Z}_{\mathbb{CQ}-4} > \mathbb{Z}_{\mathbb{CQ}-3} > \mathbb{Z}_{\mathbb{CQ}-5} > \mathbb{Z}_{\mathbb{CQ}-1} > \mathbb{Z}_{\mathbb{CQ}-2}$$

The best decision is $\mathbb{Z}_{\mathbb{CQ}-4}$ according to the technique of the CPQROFWFA operator. Further, we evaluate the comparison between proposed operators with some existing techniques to show the validity and supremacy of the proposed theory.

6. Comparative analysis

In this section, we compare the proposed ranking results with the following existing operators, such as the technique of p, q-rung orthopair fuzzy set (PQROFS) was initiated by Ibrahim and Alshammari [23] with a new condition $0 \leq \mathbb{S}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathfrak{Q}_{\mathbb{CQ}}}(\hat{x}) + \mathbb{T}_{\mathbb{Z}_{\mathbb{RP}}}^{\mathbb{O}_{\mathbb{CQ}}}(\hat{x}) \leq 1$, where the structure of QROFSs and PQROFSs are the same, but their condition are different, if $\mathfrak{Q}_{\mathbb{CQ}} = \mathbb{O}_{\mathbb{CQ}} = 1$, then the idea of PQROFSs is converted to QROFSs (a special case of the PQROFSs) [14]. Further, the novel theory of CPQROFSs was initiated by Ibrahim [25]. Comparative analysis is a major part of the manuscript and with the help of comparative analysis, we can easily determine the supremacy and validity of the proposed theory. Therefore, using the data in Table 1, the comparative analysis is listed in Table 4.

Table 4
Comparative analysis

Methods	Ranking values	Best one
Ibrahim and Alshammari [23]	Failed	Failed
Liu *et al.*, [14]	Failed	Failed
Ibrahim [25]	$\mathbb{Z}_{\mathbb{CQ}-4} > \mathbb{Z}_{\mathbb{CQ}-3} > \mathbb{Z}_{\mathbb{CQ}-5} > \mathbb{Z}_{\mathbb{CQ}-1} > \mathbb{Z}_{\mathbb{CQ}-2}$	$\mathbb{Z}_{\mathbb{CQ}-4}$
Proposed theory	$\mathbb{Z}_{\mathbb{CQ}-4} > \mathbb{Z}_{\mathbb{CQ}-3} > \mathbb{Z}_{\mathbb{CQ}-5} > \mathbb{Z}_{\mathbb{CQ}-1} > \mathbb{Z}_{\mathbb{CQ}-2}$	$\mathbb{Z}_{\mathbb{CQ}-4}$

The best decision is $\mathbb{Z}_{\mathbb{CQ}-4}$ according to the technique of the CPQROFWFA operator and also using the operators of Ibrahim [25]. Further, the existing techniques have failed because of limitations and complications, hence the proposed theory is very suitable and reliable for depicting vague and complex information in genuine life problems.

7. Conclusion

The major contribution of this article is listed below:

i. We evaluated the concept of CPQROFSs.
ii. We exposed the novel theory of fairly operational laws based on CPQROFNs.
iii. We evaluated the weighted fairly aggregation operators based on CPQROF information, called CPQROFWFA operator and CPQROFOWFA operator.
iv. We proposed some properties for the above operators, called idempotency, monotonicity, and boundedness.
v. We demonstrated the technique of MADM problems based on initiated operators.
vi. We compared our proposed ranking results with some existing techniques to show the supremacy and validity of the proposed theory.

In the future, we will extend it to circular CPQROFSs and their application in decision-making, artificial intelligence, game theory, machine learning, data science, and neural networks to enhance the worth of the proposed theory.

Acknowledgement

This research was not funded by any grant.

Conflicts of Interest

The authors declare no conflicts of interest.

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