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Solution of Fuzzy System of Linear Equation Under Different Fuzzy Difference Ideology

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ABSTRACT

Fuzzy arithmetic includes distinctive notions regarding differences between fuzzy numbers, which impacts the defining characteristics of fuzzy valued calculus. In this article, we consider Hukuhara and generalized Hukuhara differences of fuzzy numbers for solving systems of linear equations under fuzzy ruled uncertainty within an analytical discourse. Moreover, the discussion on the solutions in parametric form includes the existence and uniqueness criterion. Numerical examples and graphical representations are taken for illustrations of the proposed theory.

1. Introduction

Mathematical science has an instrumental role for modelling the natural, managerial and technological phenomena using equations and inequalities. Among them, linear equations are the most well-behaved equations. Because of its simplicity and significant characteristics, theory of linear equations has rich domain of applications in the numerous fields of sciences and technology. A system is represented by multiple unknowns and multiple constraints, the theory of the system of fuzzy linear equation necessitates. It has consequent applications on electric circuits, mechanics, economic transmission and so many disciplines.

However, it is evident that the discussions of mathematical models are mostly considered in deterministic environment. A model coming from real world phenomena may not be precisely defined by its involved variables and parameters. Uncertainty regarding data, parameters and decision process has a vital role in the model formulation and its synthesis. The Fuzzy concept introduced by Zadeh [1] in 1965 is one of the celebrated philosophies to deal uncertainty within the discourse of mathematics. To understand fuzzy concept, lots of physical scenarios are present in our

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day-to-day life. For example, temperature, for some person 27^0C is cold but for another person the temperature is 24^0C. Every person has a different feeling for hot and cold. So, how we handle this phenomenon when uncertainty depends on the prospects of an individual. In fuzzy set theory, to deal such situation, membership function is defined is different from crisp scenario.

After the introduction of the fuzzy theory, the notion was utilized and analyzed in diverging perspectives. In this part of the introduction section, we address a few significant research on linear systems using fuzzy uncertainty. In 1992, the system's consistency and all viable solutions were examined in [2]. In 1998, a system of fuzzy linear equations was explored using the embedding approach [3]. The authors solved the system using a numerical approach and proposed convergence [4]. Some writers have solved $n \times n$ systems utilising innovative strategies based on embedding methods [5]. The authors of [6] solved a system of equations with real and complex variables, considering right-side trapezoidal fuzzy numbers. In [7], the authors stated the necessary conditions for the existence and uniqueness of the solution for the fuzzy system. In [8], fuzzy systems are solved, and weak solutions are also explored. In [9] presents a generic solution to the fuzzy system. Some other writers [10-23] have provided various strategies for solving fuzzy systems. Later, the sense of fuzzy uncertainty was generalized and thus the solution approach for linear and quadratic equations was addressed in the Cauchy neotrosophic phenomena [24]. Applications of fuzzy linear and non-linear systems were manifested in the contexts of Mellin transformation [25], fractional Taylor series expansion [26], epidemic model [27]. Several discrete models [28-30] were analyzed by fuzzy difference equations, where fuzzy arithmetic especially fuzzy difference involved. On the other hand, uncertain mathematical models under continuous time frames were discussed by using fuzzy differential equation approaches [31-34].

In this article, we are interested to study the solutions of the fuzzy systems of linear equations based on the Hukuhara [35] and generalized Hukuhara [36] differences for fuzzy numbers and fuzzy valued functions. Moreover, the existence and uniqueness criteria are also examined before going for the solution of the fuzzy systems.

1.1. Objectives and Novelty

The following are some observations which motivate us to fix the objectives in this article:

i. The system of linear equation is used to formulate many mathematical models having several unknown decision variables and constraints. However, the variables and the parameters may carry uncertainty. In this sense, the system of fuzzy linear equation comes into the concerns.
ii. When a system is taken mathematically into fuzzy setting, the model should be driven by fuzzy arithmetic and calculus. It is to be mentioned that the difference between fuzzy numbers is distinguished from the crisp counterpart. So, the system of fuzzy linear equation necessities a new approach to deal such phenomena.

Based on the mentioned observation, we have the following objectives in this article:

i. To consider a system of fuzzy linear equations in Hukuhara and generalized Hukuhara sense of fuzzy difference.
ii. To check the existence and uniqueness criteria for the possible solutions.
iii. To provide a detailed analytical approach for obtaining solution of a system of linear equations.

There are lots of research articles regarding the solutions of systems of fuzzy equations. But, up to authors' knowledge, the fuzzy solution using the mentioned differences perspectives is not addressed in the literature. From this point of view, we can claim the novelty of this article.

2. Preliminaries

2.1 Fuzzy Set

A fuzzy set E of X is defined by its membership function $\mu_E : X \to [0, 1]$ which assigns to each element $x \in X$ a real number and $\mu_E(X)$ in the interval [0, 1].

Example 1: Consider a number x which is the "Numbers which are more or less 4".

So, $x = 4$ certainly, belongs to the set $\mu_E(4) = 1$. The numbers 3 and 5 belong to this set $\mu_E(3) = 0.9$, $\mu_E(3) = 0.8$. Obviously, 8 does not belong to set, $\mu_E(8) = 0$.

Figure 1 displays a fuzzy set using membership function.

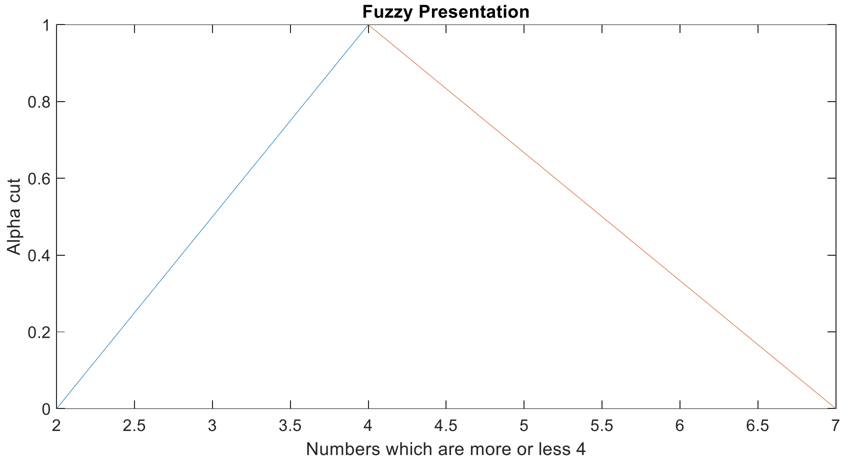

Fig. 1. The fuzzy presentation of numbers more or less 4

The support of a fuzzy set E on X denoted by supp (E) on X, is a collection of crisp numbers for which membership function is always greater than 0.
Mathematically, it is given as follows,

$$Supp(E) = \{x/x \in X, \mu_E(x) > 0\}$$

The α −cut of E is defined as,

$$^{\alpha}E = \{x \in X/\mu_E(x) \geq \alpha\}\ \forall \alpha \in [0\ 1]$$

Example 2: In Figure 1, if $\alpha = 0.4$ then $^{0.4}E = [3, 6]$, support is $^{0}E = [2, 7]$, and core is $^{1}E = \{4\}$.

A fuzzy set E is said to be Fuzzy number if it satisfies following properties,

i. The core of E is non-empty.
ii. The $\alpha - cuts$ of E are nested.
iii. The support of E is bounded.

The parametric form of fuzzy number $^{\alpha}E = [\underline{E}, \overline{E}]$

Note: The fuzzy number is denoted by $\sim$.

2.2 Fuzzy Arithmetic

Let E_1 and E_2 are the fuzzy sets. Let $^{\alpha}E_1 = [\underline{E}_1(\alpha), \overline{E}_1(\alpha)]$ and $^{\alpha}E_2 = [\underline{E}_2(\alpha), \overline{E}_2(\alpha)]$ then,
the fuzzy set operations are defined as follows,

i. $^{\alpha}E_1 \oplus {}^{\alpha}E_2 = [\underline{E}_1(\alpha), \overline{E}_1(\alpha)] \oplus [\underline{E}_2(\alpha), \overline{E}_2(\alpha)] = [\underline{E}_1(\alpha) + \underline{E}_2(\alpha), \overline{E}_1(\alpha) + \overline{E}_2(\alpha)]$
ii. $^{\alpha}E_1 \otimes {}^{\alpha}E_2 = [\underline{E}_1(\alpha), \overline{E}_1(\alpha)] \otimes [\underline{E}_2(\alpha), \overline{E}_2(\alpha)] =$
$[\min\left(\underline{E}_1(\alpha)\underline{E}_2(\alpha), \underline{E}_1(\alpha)\overline{E}_2(\alpha), \overline{E}_1(\alpha)\underline{E}_2(\alpha), \overline{E}_1(\alpha)\overline{E}_2(\alpha)\right),$

$\max\left(\underline{E_1}(\alpha)\underline{E_2}(\alpha), \underline{E_1}(\alpha)\overline{E}_2(\alpha), \overline{E}_1(\alpha)\underline{E_2}(\alpha), \overline{E}_1(\alpha)\overline{E}_2(\alpha)\right)]$

iii. $k \otimes {}^{\alpha}E = k \otimes \left[\underline{E}(\alpha), \overline{E}(\alpha)\right] = \left[k\underline{E}(\alpha), k\overline{E}(\alpha)\right]$

2.3 Fuzzy Difference

Fuzzy difference is defined by many approaches, some of popular approaches are given as follows,

Hukuhara difference:

As in [24], let $\tilde{\xi}$ and $\tilde{\eta}$ are fuzzy numbers in E then the Hukuhara difference between $\tilde{\xi}$ and $\tilde{\eta}$ in parametric form is as follows,

$$ {}^{\alpha}\tilde{\xi} \ominus_h {}^{\alpha}\tilde{\eta} = \left[\underline{\xi}(\alpha), \overline{\xi}(\alpha)\right] \ominus_h \left[\underline{\eta}(\alpha), \overline{\eta}(\alpha)\right] = \left[\underline{\xi}(\alpha) - \underline{\eta}(\alpha), \overline{\xi}(\alpha) - \overline{\eta}(\alpha)\right] $$

The Hukuhara difference exist if, length of $\tilde{\eta} \leq$ length of $\tilde{\xi}$.

Example 3: Let $\tilde{\xi} = [1, 8], \tilde{\eta} = [2, 5]$
Then $\tilde{\xi} \ominus_h \tilde{\eta} = [1, 8] \ominus_h [2, 5] = [-1, 3]$
So, $\tilde{\xi} \ominus_h \tilde{\eta}$ exists.
$\tilde{\eta} \ominus_h \tilde{\xi} = [2, 5] \ominus_h [1, 8] = [1, -3]$ (not satisfying definition of interval)
The length of $[2, 5]$ is less than equal to $[1, 8]$. Thus, the Hukuhara difference does not exist.

Generalized Hukuhara difference:

As in [25], let $\tilde{\xi}$ and $\tilde{\eta}$ are fuzzy numbers in E then the Genearlized Hukuhara difference between $\tilde{\xi}$ and $\tilde{\eta}$ is in parametric form as follows,

$$ {}^{\alpha}\tilde{\xi} \ominus_{gh} {}^{\alpha}\tilde{\eta} = \left[\underline{\xi}(\alpha), \overline{\xi}(\alpha)\right] \ominus_{gh} \left[\underline{\eta}(\alpha), \overline{\eta}(\alpha)\right] $$

$$ = \left[\min\left(\underline{\xi}(\alpha) - \underline{\eta}(\alpha), \overline{\xi}(\alpha) - \overline{\eta}(\alpha)\right), \max\left(\underline{\xi}(\alpha) - \underline{\eta}(\alpha), \overline{\xi}(\alpha) - \overline{\eta}(\alpha)\right)\right] $$

The Generalized Hukuhara difference exists for all cases.

Example 4: By considering the same value for $\tilde{\eta}$ and $\tilde{\xi}$,

$$ \tilde{\eta} \ominus_{gh} \tilde{\xi} = [2, 5] \ominus_{gh} [1, 8] = [\min(1, -3), \max(1, -3)] = [-3, 1] $$

In next section, we discuss the system of fuzzy linear equations using different fuzzy differences, Hukuhara and Generalized Hukuhara difference.

3. Analysis of System of Fuzzy Linear Equations

Consider a system of fuzzy linear equation,

$$ \tilde{A} \otimes \tilde{X} \oplus \tilde{B} = \tilde{C} \tag{1} $$

where, $\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \vdots & \vdots \\ \tilde{a}_{n1} & \cdots & \tilde{a}_{nn} \end{bmatrix}$ is $n \times n$ fuzzy matrix, $\tilde{X} = \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \end{bmatrix}$ and $\tilde{C} = \begin{bmatrix} \tilde{c}_1 \\ \vdots \\ \tilde{c}_n \end{bmatrix}$ are $n \times 1$ matrices.

Taking $\alpha - cut$ on both sides of equation (1),

$$ {}^{\alpha}\tilde{A} \otimes {}^{\alpha}\tilde{X} \oplus {}^{\alpha}\tilde{B} = {}^{\alpha}\tilde{C} $$

Which gives,

$$ \left[\underline{A}(\alpha), \overline{A}(\alpha)\right] \otimes \left[\underline{X}(\alpha), \overline{X}(\alpha)\right] \oplus \left[\underline{B}(\alpha), \overline{B}(\alpha)\right] = \left[\underline{C}(\alpha), \overline{C}(\alpha)\right] $$

Using fuzzy arithmetic as in Section 2, we get,

$$\begin{bmatrix}\min\left(\underline{A}(\alpha)\underline{X}(\alpha),\underline{A}(\alpha)\overline{X}(\alpha),\overline{A}(\alpha)\underline{X}(\alpha),\overline{A}(\alpha)\overline{X}(\alpha)\right),\\ \max\left(\underline{A}(\alpha)\underline{X}(\alpha),\underline{A}(\alpha)\overline{X}(\alpha),\overline{A}(\alpha)\underline{X}(\alpha),\overline{A}(\alpha)\overline{X}(\alpha)\right)\end{bmatrix}$$

$$= [\underline{C}(\alpha),\overline{C}(\alpha)] \ominus [\underline{B}(\alpha),\overline{B}(\alpha)] \quad (2)$$

Now, we solve equation (2), using different fuzzy differences Hukuhara and Generalized Hukuhara difference in next subsections.

3.1 Solution using Hukuhara difference concepts

Consider equation (2) under Hukuhara difference,

$$\begin{bmatrix}\min\left(\underline{A}(\alpha)\underline{X}(\alpha),\underline{A}(\alpha)\overline{X}(\alpha),\overline{A}(\alpha)\underline{X}(\alpha),\overline{A}(\alpha)\overline{X}(\alpha)\right),\\ \max\left(\underline{A}(\alpha)\underline{X}(\alpha),\underline{A}(\alpha)\overline{X}(\alpha),\overline{A}(\alpha)\underline{X}(\alpha),\overline{A}(\alpha)\overline{X}(\alpha)\right)\end{bmatrix}$$

$$= [\underline{C}(\alpha),\overline{C}(\alpha)] \ominus_h [\underline{B}(\alpha),\overline{B}(\alpha)]$$

Which gives,

$$\begin{bmatrix}\min\left(\underline{A}(\alpha)\underline{X}(\alpha),\underline{A}(\alpha)\overline{X}(\alpha),\overline{A}(\alpha)\underline{X}(\alpha),\overline{A}(\alpha)\overline{X}(\alpha)\right),\\ \max\left(\underline{A}(\alpha)\underline{X}(\alpha),\underline{A}(\alpha)\overline{X}(\alpha),\overline{A}(\alpha)\underline{X}(\alpha),\overline{A}(\alpha)\overline{X}(\alpha)\right)\end{bmatrix}$$

$$= [\underline{C}(\alpha)-\underline{B}(\alpha),\overline{C}(\alpha)-\overline{B}(\alpha)] \quad (3)$$

Comparing both sides of equation (3) component wise,

$$\min\left(\underline{A}(\alpha)\underline{X}(\alpha),\underline{A}(\alpha)\overline{X}(\alpha),\overline{A}(\alpha)\underline{X}(\alpha),\overline{A}(\alpha)\overline{X}(\alpha)\right) = \underline{C}(\alpha)-\underline{B}(\alpha) \quad (4)$$

and,

$$\max\left(\underline{A}(\alpha)\underline{X}(\alpha),\underline{A}(\alpha)\overline{X}(\alpha),\overline{A}(\alpha)\underline{X}(\alpha),\overline{A}(\alpha)\overline{X}(\alpha)\right) = \overline{C}(\alpha)-\overline{B}(\alpha) \quad (5)$$

By solving equations (4) and (5), we obtain parametric solution of equation (2).

The next section gives existence and uniqueness condition for solutions of equations (4) and (5).

Theorem 1: The solution of equations (4) and (5) exists and unique if $\underline{A}(\alpha)$ and $\overline{A}(\alpha)$ are non-singular matrices.

Proof: Consider equation (4),

$$\min\left(\underline{A}(\alpha)\underline{X}(\alpha),\underline{A}(\alpha)\overline{X}(\alpha),\overline{A}(\alpha)\underline{X}(\alpha),\overline{A}(\alpha)\overline{X}(\alpha)\right) = \underline{C}(\alpha)-\underline{B}(\alpha)$$

Let's take minimum value of equation (4) is $\underline{A}(\alpha)\underline{X}(\alpha)$ then, above equation becomes,

$$\underline{A}(\alpha)\underline{X}(\alpha) = \underline{C}(\alpha)-\underline{B}(\alpha)$$

Which gives,

$$\underline{X}(\alpha) = \frac{\underline{C}(\alpha)-\underline{B}(\alpha)}{\underline{A}(\alpha)}$$

Provided inverse of $\underline{A}(\alpha)$ must exist.

Then, the solution of equation (4) exists.

Similarly, if any other value obtained minimum from equation (4), the solution only exists if matrices $\underline{A}(\alpha)$ and $\overline{A}(\alpha)$ are non-singular.

Consider equation (5),

$$\max\left(\underline{A}(\alpha)\underline{X}(\alpha),\underline{A}(\alpha)\overline{X}(\alpha),\overline{A}(\alpha)\underline{X}(\alpha),\overline{A}(\alpha)\overline{X}(\alpha)\right) = \overline{C}(\alpha)-\overline{B}(\alpha)$$

Let's say maximum value of $\overline{A}(\alpha)\overline{X}(\alpha)$ of above equation then,

$$\overline{A}(\alpha)\overline{X}(\alpha) = \overline{C}(\alpha)-\overline{B}(\alpha)$$

i.e., $\overline{X}(\alpha) = \frac{\overline{C}(\alpha)-\overline{B}(\alpha)}{\overline{A}(\alpha)}$

Provided inverse of $\overline{A}(\alpha)$ must exist.

Similarly, if any other value obtained minimum from equation (5), the solution only exists if matrices $\underline{A}(\alpha)$ and $\overline{A}(\alpha)$ are non-singular.

For uniqueness,

Consider there are two solutions exist for equation (4) say, $\underline{X_1}(\alpha)$ and $\underline{X_2}(\alpha)$.

$$\left|\underline{X_1}(\alpha) - \underline{X_2}(\alpha)\right| \leq \left|\frac{\underline{C}(\alpha) - \underline{B}(\alpha)}{\underline{A}(\alpha)} - \frac{\underline{C}(\alpha) - \underline{B}(\alpha)}{\underline{A}(\alpha)}\right|$$

Which gives right hand side tends to 0.

Thus, $\underline{X_1}(\alpha) - \underline{X_2}(\alpha)$ tends to 0. This proves uniqueness of solution of equation (4).

Similarly, we can prove uniqueness condition for $\overline{X}(\alpha)$.

3.2 Solution using Generalized Hukuhara Difference concept

Consider equation (2) under Generalized Hukuhara difference,

$$\begin{bmatrix} \min\left(\underline{A}(\alpha)\underline{X}(\alpha), \underline{A}(\alpha)\overline{X}(\alpha), \overline{A}(\alpha)\underline{X}(\alpha), \overline{A}(\alpha)\overline{X}(\alpha)\right), \\ \max\left(\underline{A}(\alpha)\underline{X}(\alpha), \underline{A}(\alpha)\overline{X}(\alpha), \overline{A}(\alpha)\underline{X}(\alpha), \overline{A}(\alpha)\overline{X}(\alpha)\right) \end{bmatrix}$$

$$= \left[\underline{C}(\alpha), \overline{C}(\alpha)\right] \ominus_{gh} \left[\underline{B}(\alpha), \overline{B}(\alpha)\right]$$

Which gives,

$$\begin{bmatrix} \min\left(\underline{A}(\alpha)\underline{X}(\alpha), \underline{A}(\alpha)\overline{X}(\alpha), \overline{A}(\alpha)\underline{X}(\alpha), \overline{A}(\alpha)\overline{X}(\alpha)\right), \\ \max\left(\underline{A}(\alpha)\underline{X}(\alpha), \underline{A}(\alpha)\overline{X}(\alpha), \overline{A}(\alpha)\underline{X}(\alpha), \overline{A}(\alpha)\overline{X}(\alpha)\right) \end{bmatrix}$$

$$= \left[\min\left(\underline{C}(\alpha) - \underline{B}(\alpha), \overline{C}(\alpha) - \overline{B}(\alpha)\right), \max(\underline{C}(\alpha) - \underline{B}(\alpha), \overline{C}(\alpha) - \overline{B}(\alpha))\right] \tag{6}$$

Comparing component wise of equation (6),

$$\min\left(\underline{A}(\alpha)\underline{X}(\alpha), \underline{A}(\alpha)\overline{X}(\alpha), \overline{A}(\alpha)\underline{X}(\alpha), \overline{A}(\alpha)\overline{X}(\alpha)\right) = \min\left(\underline{C}(\alpha) - \underline{B}(\alpha), \overline{C}(\alpha) - \overline{B}(\alpha)\right) \tag{7}$$

and

$$\max\left(\underline{A}(\alpha)\underline{X}(\alpha), \underline{A}(\alpha)\overline{X}(\alpha), \overline{A}(\alpha)\underline{X}(\alpha), \overline{A}(\alpha)\overline{X}(\alpha)\right) = \max(\underline{C}(\alpha) - \underline{B}(\alpha), \overline{C}(\alpha) - \overline{B}(\alpha)) \tag{8}$$

The possibilities for equations (7) and (8) are as follows,

$$\underline{A}(\alpha)\underline{X}(\alpha) = \underline{C}(\alpha) - \underline{B}(\alpha)$$
$$\underline{A}(\alpha)\overline{X}(\alpha) = \underline{C}(\alpha) - \underline{B}(\alpha)$$
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$$\overline{A}(\alpha)\overline{X}(\alpha) = \overline{C}(\alpha) - \overline{B}(\alpha)$$

The solution of equations (7) and (8) exists and unique iff matrices $\underline{A}(\alpha)$ and $\overline{A}(\alpha)$ are non-singular.

Note: The proof is similar in Subsection 3.1.

Remark: Consider the system $\tilde{A} \otimes \tilde{X} = \tilde{B}$. This system gives more appropriate solution under Hukuhara difference.

Next section solves numerical example under proposed method

4. Numerical illustration

Consider a simple 2×2 fuzzy system of linear equations as follows,

$$\tilde{x} \oplus \tilde{y} = \widetilde{10}$$
$$\tilde{x} \ominus \tilde{y} = \tilde{5}$$

First, we take $\alpha -$ cut, on above equations,

$$^{\alpha}\tilde{x} \oplus {}^{\alpha}\tilde{y} = {}^{\alpha}\widetilde{10} \quad (9)$$
$$^{\alpha}\tilde{x} \ominus {}^{\alpha}\tilde{y} = {}^{\alpha}\tilde{5} \quad (10)$$

Where $^{\alpha}\widetilde{10} = [9 + \alpha, 11 - \alpha]$ and $^{\alpha}\tilde{5} = [4 + \alpha, 6 - \alpha]$

Writing parametric form of equation (9),

$$\left[\underline{x}(\alpha), \overline{x}(\alpha)\right] \oplus \left[\underline{y}(\alpha), \overline{y}(\alpha)\right] = [9 + \alpha, 11 - \alpha]$$

Using fuzzy arithmetic, we have,

$$\left[\underline{x}(\alpha) + \underline{y}(\alpha), \overline{x}(\alpha) + \overline{y}(\alpha)\right] = [9 + \alpha, 11 - \alpha]$$

Which gives,

$$\underline{x}(\alpha) + \underline{y}(\alpha) = 9 + \alpha \quad (11)$$
$$\overline{x}(\alpha) + \overline{y}(\alpha) = 11 - \alpha \quad (12)$$

For equation (10), we have,

$$\left[\underline{x}(\alpha), \overline{x}(\alpha)\right] \ominus \left[\underline{y}(\alpha), \overline{y}(\alpha)\right] = [9 + \alpha, 11 - \alpha]$$

Simplify equation (10) under both differences, Hukuhara difference and generalized Hukuhara difference.

4.1 Under Hukuhara difference:

$$\left[\underline{x}(\alpha) - \underline{y}(\alpha), \overline{x}(\alpha) - \overline{y}(\alpha)\right] = [4 + \alpha, 6 - \alpha]$$

Comparing component wise, we have,

$$\underline{x}(\alpha) - \underline{y}(\alpha) = 4 + \alpha \quad (13)$$
$$\overline{x}(\alpha) - \overline{y}(\alpha) = 6 - \alpha \quad (14)$$

From equations (11) and (13),

$$\underline{x}(\alpha) = 6.5 + \alpha, \underline{y}(\alpha) = 2.5$$

From equations (12) and (14),

$$\overline{x}(\alpha) = 8.5 - \alpha, \overline{y}(\alpha) = 2.5$$

The fuzzy solution is displayed graphically in Figure 2.

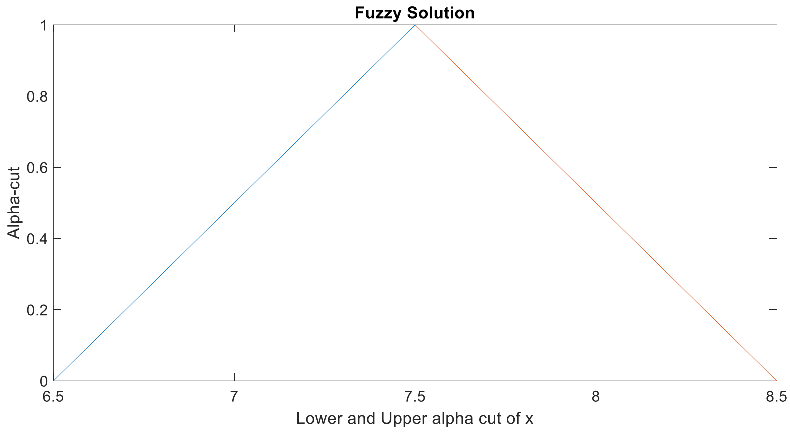

Fig. 2. Fuzzy representation

Obtained solution is fuzzy and also matches with crisp solution of the given problem in crisp form.

4.2 Under generalized Hukuhara difference:

$$\left[min\left(\underline{x}(\alpha)-\underline{y}(\alpha),\overline{x}(\alpha)-\overline{y}(\alpha)\right),\max\left(\underline{x}(\alpha)-\underline{y}(\alpha),\overline{x}(\alpha)-\overline{y}(\alpha)\right)\right]=[4+\alpha,6-\alpha]$$

Comparing component wise, we have,

$$min\left(\underline{x}(\alpha)-\underline{y}(\alpha),\overline{x}(\alpha)-\overline{y}(\alpha)\right)=4+\alpha$$

$$\max\left(\underline{x}(\alpha)-\underline{y}(\alpha),\overline{x}(\alpha)-\overline{y}(\alpha)\right)=6-\alpha$$

Possibilities are,

$$\underline{x}(\alpha)-\underline{y}(\alpha)=4+\alpha$$
$$\overline{x}(\alpha)-\overline{y}(\alpha)=4+\alpha$$
$$\underline{x}(\alpha)-\underline{y}(\alpha)=6-\alpha$$
$$\overline{x}(\alpha)-\overline{y}(\alpha)=6-\alpha$$

For solution, we have following possibilities,

i. $\underline{x}(\alpha)+\underline{y}(\alpha)=9+\alpha$, $(\underline{x}(\alpha)-\underline{y}(\alpha)=4+\alpha$ and $\underline{x}(\alpha)-\underline{y}(\alpha)=6-\alpha)$

Which gives two sets of possible solution, $\underline{x}(\alpha)=7.5,\underline{y}(\alpha)=1.5+\alpha$ and $\underline{x}(\alpha)=6.5+\ \alpha,\underline{y}(\alpha)=2.5$

ii. $\overline{x}(\alpha)+\overline{y}(\alpha)=11-\alpha$, $(\overline{x}(\alpha)-\overline{y}(\alpha)=4+\alpha$ and $\overline{x}(\alpha)-\overline{y}(\alpha)=6-\alpha)$

Which gives two sets of possible solutions,

$\overline{x}(\alpha)=8.5-\alpha,\overline{y}(\alpha)=2.5$ and $\overline{x}(\alpha)=7.5,\overline{y}(\alpha)=3.5-\alpha$

The solution under generalized Hukuhara difference for, $\underline{x}(\alpha)=7.5$ and $\underline{x}(\alpha)=6.5+\ \alpha$ is graphically displayed in Figure 3.

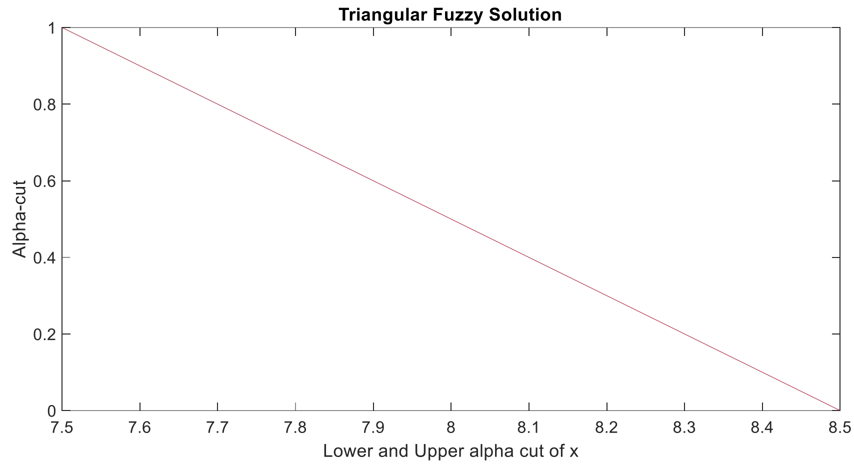

Fig. 3. Triangular Fuzzy Representation for x

The solution under generalized Hukuhara difference for $\underline{y}(\alpha) = 1.5 + \alpha$ and $\overline{y}(\alpha) = 2.5$ is graphically displayed in Figure 4.

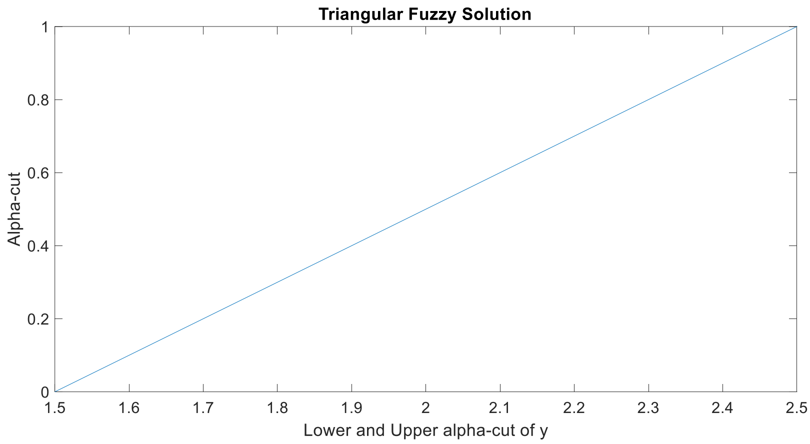

Fig. 4. Triangular Fuzzy Representation for y

Although the obtained solution is fuzzy in Figures.3 and 4 but it does not match with crisp solution of the given problem in crisp form.

5. Results and Discussion

The solution obtained from Hukuhara difference is $\underline{x}(\alpha) = 6.5 + \alpha, \underline{y}(\alpha) = 2.5$ and $\overline{x}(\alpha) = 8.5 - \alpha, \overline{y}(\alpha) = 2.5$. At $\alpha = 1$, the solution matches with crisp solution of the given problem in crisp form. Under generalized Hukuhara difference, there are two possibilities and in both obtained solution in fuzzy. But only one solution matches with crisp solution of given problem in crisp form. For particular system, $\tilde{A} \otimes \tilde{X} = \tilde{B}$, Hukuhara difference is the appropriate difference which gives fuzzy solution that matches with crisp solution at core.

6. Conclusion

In this paper, the systems of fuzzy linear equations have been taken into consideration for analysing and finding the possible solutions. We have used Hukuhara and generalized Hukuhara differences for tackling the fuzzy arithmetic involved in the system. The existence and uniqueness of the solutions have been checked before obtaining the solutions in this article. It is perceived here that Hukuhara differences is fitted for a few special cases. However, generalized Hukuhara difference is a more impacting notion regarding fuzzy arithmetic and calculus because in the later definition, the fuzzy difference exists, irrespective of the cases. There are several real-world managerial examples which include multiple unknowns and constraints in an uncertain strategic decision phenomenon. Such phenomena should be the field of application of the proposed theory. The present article can also be extended by introducing the numerical methods for arbitrary polynomial and transcendental equations under fuzzy systems with Hukuhara and generalized Hukuhara differences in future

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Conflicts of Interest
The authors declare no conflicts of interest.

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