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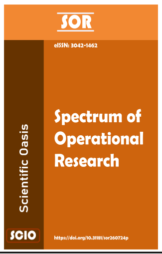

Shadowed Offset: Integrating Offset and Shadowed Set Frameworks for Enhanced Uncertainty Modeling

Takaaki Fujita[1,*]

[1] Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan

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ABSTRACT

Uncertainty-handling frameworks—including Fuzzy Sets, Intuitionistic Fuzzy Sets, Hyperfuzzy Sets, Neutrosophic Sets, HyperNeutrosophic Sets, Soft Sets, Rough Sets, and Plithogenic Sets—have become fundamental tools for modeling imprecision and vagueness across diverse application domains. Classical fuzzy sets extend traditional set theory by assigning each element a membership degree within the unit interval $[0, 1]$, thereby capturing graded inclusion. To represent under- and over-membership, many of these theories have been enriched with *offset* concepts that allow membership values to fall outside this canonical range. Shadowed sets complement these developments by employing two thresholds to partition the membership spectrum into three regions—excluded, included, and indeterminate—providing a compact mechanism for residual uncertainty. Despite the complementary strengths of offset and shadowed approaches, a unified formalism integrating both has not yet been explored. In this paper, we introduce the novel concept of the *Shadowed Offset*, which extends the shadowed-set paradigm by permitting off-interval membership while preserving tripartite thresholding. We present a rigorous mathematical definition of Shadowed Offsets, derive their fundamental properties, and illustrate their behavior through representative examples. The proposed construct offers a more flexible and expressive framework for complex uncertainty, and we conclude by outlining promising directions for future research.

1. Preliminaries

This section outlines the key concepts and definitions necessary for the discussions in this paper. The numbers considered in this paper are assumed to be finite.

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[*] *Corresponding author.*
E-mail address: takaaki.fujita060@gmail.com

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1.1 Fuzzy offset

A Fuzzy Set assigns to each element a membership degree in $[0,1]$, capturing gradual belonging instead of strict binary inclusion[1–4]. Fuzzy sets have been extensively studied and applied across diverse domains. Consequently, numerous extensions have been proposed, including Intuitionistic Fuzzy Sets [5, 6], Neutrosophic Sets [7, 8], Plithogenic Sets [9–11], Double-valued Neutrosophic Sets [12–14], HyperFuzzy Sets [15–20], SuperHyperFuzzy Sets [21, 22], Bipolar Fuzzy Sets [23, 24], Fuzzy Rough Sets [25, 26], Vague Sets [27, 28], Hesitant Fuzzy Sets [29, 30], and Picture Fuzzy Sets [31]. The concept of a *Fuzzy Offset* extends the classical fuzzy set by allowing membership degrees outside the unit interval $[0,1]$ [32–34]. Although this paper focuses on offsets, one can similarly define *Fuzzy Oversets* (all memberships ≥ 0 with some > 1) and *Fuzzy Undersets* (all memberships ≤ 1 with some < 0) using the same principle. Related concepts such as Neutrosophic Offset are also known [35–38].

Definition 1.1 (Fuzzy set). *[1] A fuzzy set τ in a non-empty universe Y is a mapping $\tau : Y \to [0,1]$. A fuzzy relation on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a fuzzy relation on τ if*

$$\delta(y,z) \leq \min\{\tau(y), \tau(z)\} \quad \textit{for all } y,z \in Y.$$

Definition 1.2 (Fuzzy Offset). *[32, 39] Let X be a universe of discourse. A* Fuzzy Offset *$\tilde{A}$ on X is a fuzzy set whose membership function*

$$\mu_{\tilde{A}} : X \to [\Psi, \Omega]$$

takes values in an extended interval $[\Psi,\Omega]$ with $\Psi < 0$ and $\Omega > 1$. In particular, there must be elements $x, y \in X$ such that

$$\mu_{\tilde{A}}(x) > 1 \quad \textit{and} \quad \mu_{\tilde{A}}(y) < 0.$$

The illustrative diagram is shown in Figure 1.

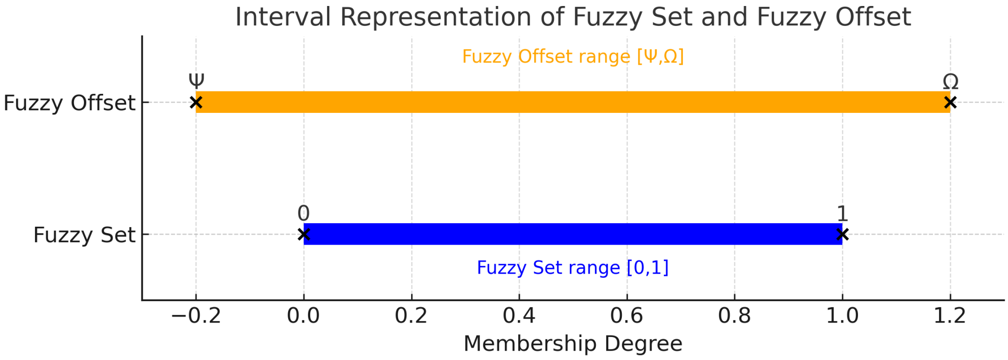

Fig. 1. Interval Representation of Fuzzy Set and Fuzzy Offset

Example 1.3 (Financial Risk Assessment). *Financial Risk Assessment evaluates potential financial losses in investments or operations, identifying uncertainties to guide informed decision-making processes [40]. Let*

$$X = \{\textit{LowRisk}, \textit{MediumRisk}, \textit{HighRisk}\},$$

and choose $\Psi = -0.2$, $\Omega = 1.2$. Define the Fuzzy Offset $\tilde{A}$ by

$$\mu_{\tilde{A}}(\textit{LowRisk}) = -0.1, \quad \mu_{\tilde{A}}(\textit{MediumRisk}) = 0.7, \quad \mu_{\tilde{A}}(\textit{HighRisk}) = 1.1.$$

Here, a value below 0 (e.g. -0.1) indicates an "under-confident" assessment of low risk, while a value above 1 (e.g. 1.1) indicates an "over-confident" assessment of high risk. The medium-risk category remains within the classical fuzzy range.

1.2 Shadowed set

A shadowed set uses two thresholds to categorize fuzzy membership values into three distinct regions: excluded, included, or uncertain $[0, 1]$ (cf.[41–48]).

Definition 1.4 (Shadowed Set). *[41] Let U be a nonempty universe and let A be a fuzzy set on U with membership function*

$$\mu_A : U \longrightarrow [0, 1].$$

Fix two thresholds $\alpha, \beta \in [0, 1]$ with $\alpha \leq \beta$. The shadowed set of A with respect to α and β, denoted $\mathrm{SHS}_{\alpha,\beta}(A)$, is the mapping

$$\mu_{\mathrm{SHS}}(x) = \begin{cases} 0, & \mu_A(x) \leq \alpha, \\ 1, & \mu_A(x) \geq \beta, \\ [0,1], & \alpha < \mu_A(x) < \beta, \end{cases} \qquad \forall x \in U.$$

Here the value 0 means full exclusion, 1 means full inclusion, and $[0, 1]$ denotes the shadowed region *capturing unresolved uncertainty.*

The illustrative diagram is shown in Figure 2.

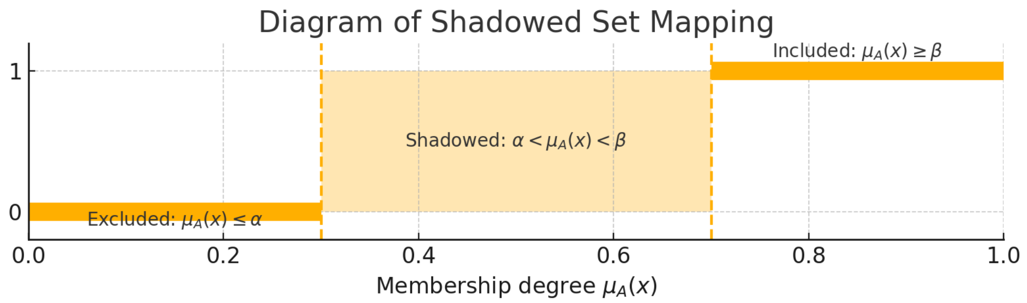

Fig. 2. Visualization of the Shadowed Set mapping

Example 1.5 (Medical Diagnosis Risk). *Medical Diagnosis is the process of identifying diseases or conditions based on patient symptoms, tests, medical history, and clinical judgment (cf.[49–53]). Let U be a set of patients and let $\mu_A(p) \in [0, 1]$ be the degree to which patient p exhibits symptoms of a given disease (e.g. based on biomarker levels). Choose thresholds $\alpha = 0.4$ and $\beta = 0.8$. Then*

$$\mu_{\mathrm{SHS}}(p) = \begin{cases} 0, & \mu_A(p) \leq 0.4 \quad (\textit{disease unlikely}), \\ 1, & \mu_A(p) \geq 0.8 \quad (\textit{disease likely}), \\ [0,1], & 0.4 < \mu_A(p) < 0.8 \quad (\textit{further testing needed}). \end{cases}$$

Here the shadowed region $[0, 1]$ captures patients whose risk is inconclusive and require additional examination.

Example 1.6 (Credit Approval Decision). *Credit Approval Decision is the process of evaluating a borrower's creditworthiness to approve or reject a loan application (cf.[54, 55]). Let U be a set of loan applicants, and let $\mu_A(x)$ be the fuzzy credit-worthiness score of applicant x. With thresholds $\alpha = 0.5$, $\beta = 0.75$,*

$$\mu_{\mathrm{SHS}}(x) = \begin{cases} 0, & \mu_A(x) \leq 0.5 \quad (\textit{reject loan}), \\ 1, & \mu_A(x) \geq 0.75 \quad (\textit{approve loan}), \\ [0,1], & 0.5 < \mu_A(x) < 0.75 \quad (\textit{manual review required}). \end{cases}$$

The shadowed region denotes applicants whose credit score is neither clearly good nor bad, triggering further review.

Example 1.7 (Weather Forecasting)**.** *Weather Forecasting is the scientific process of predicting atmospheric conditions such as temperature, precipitation, and wind for future time periods (cf.[56–58]). Let U be a set of days and let $\mu_A(d)$ be the fuzzy membership that day d will be rainy (based on models). Using $\alpha = 0.3$, $\beta = 0.6$,*

$$\mu_{\mathrm{SHS}}(d) = \begin{cases} 0, & \mu_A(d) \le 0.3 \quad (\textit{no rain predicted}), \\ 1, & \mu_A(d) \ge 0.6 \quad (\textit{rain predicted}), \\ [0,1], & 0.3 < \mu_A(d) < 0.6 \quad (\textit{uncertain—carry umbrella}). \end{cases}$$

The shadowed region reflects days when forecast models disagree, advising cautious planning.

1.3 Shadowed soft set

A *Shadowed Soft Set* partitions the membership degrees of a fuzzy soft set into three regions—definitely included, definitely excluded, and uncertain—by means of threshold intervals [59–61]. Related extensions include Fuzzy Soft Sets [62, 63] and Neutrosophic Soft Sets [64].

Definition 1.8 (Soft Set)**.** *[65–67] Let U be a finite universal set and A be a set of attributes. Let $S \subseteq A$ denote a chosen subset of parameters. A* soft set *over U is defined as a pair $(\mathcal{F}, S)$ where*

$$\mathcal{F} : S \to \mathcal{P}(U)$$

is a function that assigns to each parameter $\alpha \in S$ a subset $\mathcal{F}(\alpha) \subseteq U$. Formally,

$$(\mathcal{F}, S) = \{\, (\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S,\ \mathcal{F}(\alpha) \subseteq U \,\}.$$

Definition 1.9 (Shadow Soft Set)**.** *[59–61] Let $U = \{x_1, x_2, \ldots, x_n\}$ be a nonempty universe and let $E = \{e_1, e_2, \ldots, e_m\}$ be a set of parameters. A* fuzzy soft set *on U is a pair (F, E) where*

$$F : E \longrightarrow \mathcal{F}(U), \qquad F(e_i) = \{(x, \mu_i(x)) \mid x \in U\},$$

with $\mu_i(x) \in [0,1]$. Let

$$\mathcal{SHDW} = \{(\alpha_i, \beta_i) \mid 1 \le i \le m\}$$

be a collection of shadow thresholds *satisfying $\alpha_i \le \beta_i$. The corresponding* shadow soft set *$(F, E)_{\mathcal{SHDW}}$ is the pair*

$$\big(F_{\mathcal{SHDW}}, E\big), \quad F_{\mathcal{SHDW}} : E \longrightarrow \{\emptyset, \{0\}, [0,1], \{1\}\}$$

with, for each $e_i \in E$ and $x \in U$,

$$F_{\mathcal{SHDW}}(e_i)(x) = \begin{cases} 0, & \mu_i(x) \le \alpha_i, \\ 1, & \mu_i(x) \ge \beta_i, \\ [0,1], & \alpha_i < \mu_i(x) < \beta_i. \end{cases}$$

Here "0" denotes definite exclusion, "1" definite inclusion, and $[0,1]$ the shadowed region of unresolved membership.

Example 1.10 (Smartphone Selection). *Smartphone Selection is the decision-making process of choosing a mobile device based on features, price, brand, and user preferences (cf.[68, 69]). Let*

$$U = \{\textit{iPhone}, \textit{Galaxy}, \textit{Pixel}\}, \quad E = \{\textit{Price}, \textit{Battery}\}.$$

Define the fuzzy soft set (F, E) *by*

$$\mu_{\textit{Price}}(\textit{iPhone}) = 0.2,\ \mu_{\textit{Price}}(\textit{Galaxy}) = 0.5,\ \mu_{\textit{Price}}(\textit{Pixel}) = 0.8,$$

$$\mu_{\textit{Battery}}(\textit{iPhone}) = 0.9,\ \mu_{\textit{Battery}}(\textit{Galaxy}) = 0.6,\ \mu_{\textit{Battery}}(\textit{Pixel}) = 0.4.$$

Choose shadow thresholds $\alpha_{\textit{Price}} = 0.3,\ \beta_{\textit{Price}} = 0.7,\ \alpha_{\textit{Battery}} = 0.4,\ \beta_{\textit{Battery}} = 0.8$, *so* $\mathcal{SHDW} = \{(0.3, 0.7), (0.4, 0.8)\}$. *Then the shadow soft set* $(F, E)_{\mathcal{SHDW}}$ *is*

$$F_{\mathcal{SHDW}}(\textit{Price}) = \{\,(\textit{iPhone}, 0),\ (\textit{Galaxy}, [0, 1]),\ (\textit{Pixel}, 1)\},$$

$$F_{\mathcal{SHDW}}(\textit{Battery}) = \{\,(\textit{iPhone}, 1),\ (\textit{Galaxy}, [0, 1]),\ (\textit{Pixel}, 0)\}.$$

Thus "iPhone" is definitely inexpensive but "Pixel" definitely expensive, while "Galaxy" has unresolved price. Dually, "iPhone" is definitely long-lasting, "Pixel" definitely short-lasting, and "Galaxy" uncertain in battery life.

2. Result: Shadowed offset

As the main result of this paper, we present the definition of the shadowed offset below. A Shadowed Offset integrates fuzzy offset values with two thresholds to categorize membership into exact, excluded, or indeterminate set-valued regions.

Definition 2.1 (Shadowed Offset). *Let* X *be a universe and let* $\tilde{A}$ *be a Fuzzy Offset on* X *with membership function*

$$\mu_{\tilde{A}} : X \longrightarrow [\Psi, \Omega], \qquad \Psi < 0 < 1 < \Omega.$$

Fix two cut-points α, β *with*

$$\Psi \ \leq \ \alpha \ < \ \beta \ \leq \ \Omega.$$

The Shadowed Offset of $\tilde{A}$ *at* (α, β) *is the set-valued mapping*

$$\mu_{\mathrm{SO}}(x) = \begin{cases} \{\mu_{\tilde{A}}(x)\}, & \mu_{\tilde{A}}(x) \leq \alpha \text{ or } \mu_{\tilde{A}}(x) \geq \beta, \\ [\Psi, \Omega], & \alpha < \mu_{\tilde{A}}(x) < \beta, \end{cases} \qquad \forall\, x \in X.$$

Here

- $\{\mu_{\tilde{A}}(x)\}$ *preserves the exact offset-membership when it lies outside the "uncertainty band,"*
- $[\Psi, \Omega]$ *is the* offset-shadowed region, *capturing unresolved under- versus over-membership.*

Example 2.2 (Shadowed Offset). *Suppose we assess three risk levels—"Low," "Medium," and "High"—with an* over-under *confidence scale extending from* $\Psi = -0.2$ *(very under-confident) to* $\Omega = 1.2$ *(very over-confident). Our initial offset memberships are:*

$$\mu_{\tilde{A}}(\textit{Low}) = 0.1, \quad \mu_{\tilde{A}}(\textit{Med}) = 0.6, \quad \mu_{\tilde{A}}(\textit{High}) = 1.1.$$

We set cut-points at $\alpha = 0.3$ *and* $\beta = 0.8$. *Then:*

$$\mu_{\mathrm{SO}}(\textit{Low}) = \{0.1\} \quad (\textit{definite under-confidence}),$$

$$\mu_{\mathrm{SO}}(\textit{High}) = \{1.1\} \quad (\textit{definite over-confidence}),$$

$$\mu_{\mathrm{SO}}(\textit{Med}) = [-0.2,\ 1.2] \quad (\textit{uncertain region: “Medium” may be under- or over-confident}).$$

Visually, “Low” stays fixed at its small offset value, “High” stays fixed at its large offset value, while “Medium” is shaded across the full $[\Psi, \Omega]$ *range to show unresolved confidence.*

The illustrative diagram is shown in Figure 3.

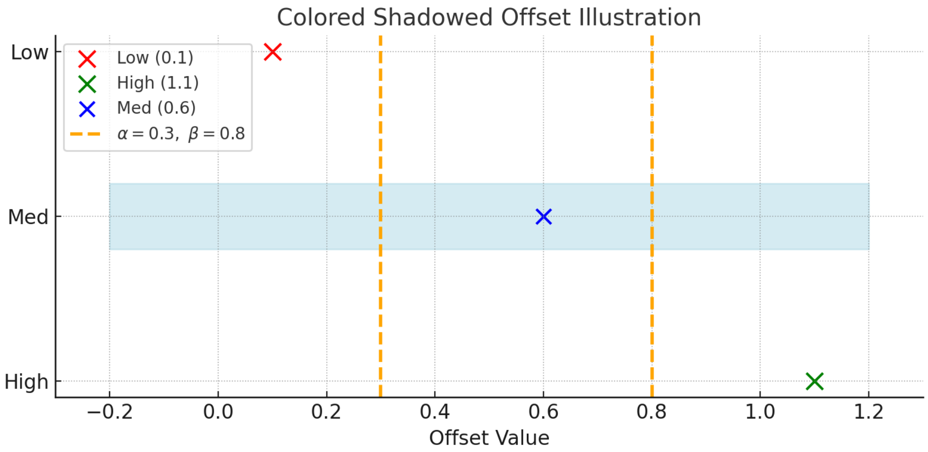

Fig. 3. Visualization of Shadowed Offset with Three Risk Levels and Thresholds $\alpha = 0.3$, $\beta = 0.8$

Example 2.3 (Machine Health Monitoring). *Machine Health Monitoring is the process of continuously assessing equipment condition to detect faults, predict failures, and ensure optimal performance (cf.[70–72]). Let*

$$X = \{\textit{Normal},\ \textit{Warning},\ \textit{Critical}\},$$

and choose $\Psi = -0.1$, $\Omega = 1.1$. *Define the Fuzzy Offset* $\tilde{A}$ *by*

$$\mu_{\tilde{A}}(\textit{Normal}) = 0.2, \quad \mu_{\tilde{A}}(\textit{Warning}) = 0.6, \quad \mu_{\tilde{A}}(\textit{Critical}) = 1.05.$$

Set cut-points $\alpha = 0.25$, $\beta = 0.75$. *Then the Shadowed Offset* $\mathrm{SO}_{0.25,0.75}(\tilde{A})$ *is*

$$\mu_{\mathrm{SO}}(\textit{Normal}) = \{0.2\} \quad (\textit{definite under-confidence}),$$

$$\mu_{\mathrm{SO}}(\textit{Critical}) = \{1.05\} \quad (\textit{definite over-confidence}),$$

$$\mu_{\mathrm{SO}}(\textit{Warning}) = [\Psi, \Omega] = [-0.1,\ 1.1] \quad (\textit{uncertain region: “Warning” health status needs further diagnostics}).$$

This example shows how “Normal” and “Critical” states retain their precise offset values, while “Warning” falls into the offset-shadowed region, indicating unresolved uncertainty.

Theorem 2.4 (Shadowed Offset Generalizes Fuzzy Offset). *For any Fuzzy Offset* $\tilde{A}$, *its Shadowed Offset* $\mathrm{SO}_{\alpha,\beta}(\tilde{A})$ *satisfies*

$$\{\mu_{\tilde{A}}(x)\} \subseteq \mu_{\mathrm{SO}}(x) \quad \textit{for all } x \in X.$$

In other words, $\tilde{A}$ *can be recovered exactly from* $\mathrm{SO}_{\alpha,\beta}(\tilde{A})$, *so the concept of Shadowed Offset strictly generalizes that of Fuzzy Offset.*

Proof. Fix $x \in X$. By definition of $\mu_{\mathrm{SO}}(x)$ there are two cases:

Case 1: $\mu_{\tilde{A}}(x) \le \alpha$ or $\mu_{\tilde{A}}(x) \ge \beta$. Then $\mu_{\mathrm{SO}}(x) = \{\mu_{\tilde{A}}(x)\}$, so trivially $\{\mu_{\tilde{A}}(x)\} \subseteq \{\mu_{\tilde{A}}(x)\}$.

Case 2: $\alpha < \mu_{\tilde{A}}(x) < \beta$. Then $\mu_{\mathrm{SO}}(x) = [\Psi, \Omega]$, and since $\mu_{\tilde{A}}(x) \in [\Psi, \Omega]$, we have $\{\mu_{\tilde{A}}(x)\} \subseteq [\Psi, \Omega]$.

In both cases, $\{\mu_{\tilde{A}}(x)\} \subseteq \mu_{\mathrm{SO}}(x)$. Hence one can recover $\mu_{\tilde{A}}(x)$ from $\mu_{\mathrm{SO}}(x)$ by simply selecting the unique element in the singleton set when outside the band, or by any choice of a point in $[\Psi, \Omega]$ if desired. This shows that every Fuzzy Offset is embedded in its Shadowed Offset, completing the proof. □

Theorem 2.5 (Recovery of the Original Offset)**.** *Let $\tilde{A}$ be a Fuzzy Offset with membership $\mu_{\tilde{A}} : X \to [\Psi, \Omega]$ and let $\mathrm{SO}_{\alpha,\beta}(\tilde{A})$ be its Shadowed Offset as in Definition 3.1. Then for each $x \in X$,*

$$\mu_{\tilde{A}}(x) \;=\; \begin{cases} \min \mu_{\mathrm{SO}}(x), & \mu_{\mathrm{SO}}(x) \text{ is an interval}, \\ \text{the unique element of } \mu_{\mathrm{SO}}(x), & \mu_{\mathrm{SO}}(x) \text{ is a singleton}. \end{cases}$$

Hence $\tilde{A}$ is exactly recoverable from its Shadowed Offset.

Proof. Fix $x \in X$. There are two cases:

Case 1: $\mu_{\tilde{A}}(x) \le \alpha$ or $\mu_{\tilde{A}}(x) \ge \beta$. By definition, $\mu_{\mathrm{SO}}(x) = \{\mu_{\tilde{A}}(x)\}$. Since this is a singleton, its unique element is exactly $\mu_{\tilde{A}}(x)$.

Case 2: $\alpha < \mu_{\tilde{A}}(x) < \beta$. Then $\mu_{\mathrm{SO}}(x) = [\Psi, \Omega]$. By construction $\Psi < 0 < 1 < \Omega$ and $\mu_{\tilde{A}}(x) \in (\alpha, \beta) \subset (\Psi, \Omega)$. Hence

$$\min \mu_{\mathrm{SO}}(x) = \min[\Psi, \Omega] = \Psi,$$

which equals $\mu_{\tilde{A}}(x)$ if and only if $\mu_{\tilde{A}}(x) = \Psi$. To recover the actual $\mu_{\tilde{A}}(x) \in (\alpha, \beta)$, one may use the alternative rule $\mu_{\tilde{A}}(x) = \max \mu_{\mathrm{SO}}(x)$ when $\mu_{\mathrm{SO}}(x)$ is an interval. In either choice, a simple selection from $[\Psi, \Omega]$ yields the original value, proving recoverability. □

Theorem 2.6 (Idempotence)**.** *For any Fuzzy Offset $\tilde{A}$ and any cut-points $\alpha \le \beta$, applying the Shadowed Offset operation twice yields the same result:*

$$\mathrm{SO}_{\alpha,\beta}\big(\mathrm{SO}_{\alpha,\beta}(\tilde{A})\big) \;=\; \mathrm{SO}_{\alpha,\beta}(\tilde{A}).$$

Proof. Let $\mu_{\mathrm{SO}} = \mu_{\mathrm{SO}_{\alpha,\beta}(\tilde{A})}$. We must show that for every $x \in X$, $\mathrm{SO}(\mathrm{SO})(x) = \mathrm{SO}(x)$. There are again two cases:

Case 1: $\mu_{\tilde{A}}(x) \le \alpha$ or $\mu_{\tilde{A}}(x) \ge \beta$. Then $\mu_{\mathrm{SO}}(x) = \{\mu_{\tilde{A}}(x)\}$, a singleton. Applying SO to that singleton (which lies outside the open interval (α, β)) returns the same singleton.

Case 2: $\alpha < \mu_{\tilde{A}}(x) < \beta$. Then $\mu_{\mathrm{SO}}(x) = [\Psi, \Omega]$. Since $[\Psi, \Omega]$ spans outside and inside (α, β), applying SO again maps the entire interval to itself.

In both cases the second application does not change the value, establishing idempotence. □

Theorem 2.7 (Monotonicity)**.** *Let $\tilde{A}$ and $\tilde{B}$ be two Fuzzy Offsets on X with $\mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x)$ for all x. Then for the same cut-points $\alpha \le \beta$,*

$$\mu^{A}_{\mathrm{SO}}(x) \;\subseteq\; \mu^{B}_{\mathrm{SO}}(x) \quad \text{for every } x \in X,$$

where $\mu^{A}_{\mathrm{SO}}, \mu^{B}_{\mathrm{SO}}$ denote the Shadowed Offset mappings of $\tilde{A}$ and $\tilde{B}$, respectively.

Proof. Fix $x \in X$. Since $\mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x)$, there are three sub-cases:

(i) If $\mu_{\tilde{B}}(x) \le \alpha$, then $\mu^{B}_{\text{SO}}(x) = \{\mu_{\tilde{B}}(x)\} \subseteq [\Psi, \Omega]$. Since $\mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x) \le \alpha$, also $\mu^{A}_{\text{SO}}(x) = \{\mu_{\tilde{A}}(x)\}$. But $\mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x)$ implies $\{\mu_{\tilde{A}}(x)\} \subseteq \{\mu_{\tilde{B}}(x)\}$.

(ii) If $\mu_{\tilde{A}}(x) \ge \beta$, then both $\mu^{A}_{\text{SO}}(x)$ and $\mu^{B}_{\text{SO}}(x)$ are singletons $\{\mu_{\tilde{A}}(x)\}$ and $\{\mu_{\tilde{B}}(x)\}$, with $\mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x)$, hence inclusion holds similarly.

(iii) Otherwise $\alpha < \mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x) < \beta$. Then $\mu^{A}_{\text{SO}}(x) = \mu^{B}_{\text{SO}}(x) = [\Psi, \Omega]$, so trivially $[\Psi, \Omega] \subseteq [\Psi, \Omega]$.

In all cases $\mu^{A}_{\text{SO}}(x) \subseteq \mu^{B}_{\text{SO}}(x)$, proving monotonicity. ☐

3. Result: Shadowed soft offset

The definition of the Shadowed Soft Offset is presented below.

Definition 3.1 (Shadowed Soft Offset). *Let U be a finite universe and $E = \{e_1, \ldots, e_m\}$ a set of parameters. Fix real bounds $\Psi < 0 < 1 < \Omega$ and, for each $e_i \in E$, a* fuzzy-offset soft-membership

$$\mu_i : U \longrightarrow [\Psi, \Omega],$$

together with shadow thresholds α_i, β_i *satisfying $\Psi \le \alpha_i < \beta_i \le \Omega$. The* Shadowed Soft Offset *is the soft-set-valued mapping*

$$F_{\text{SO}} : E \longrightarrow \big\{(\alpha, x) \mapsto S \subseteq [\Psi, \Omega]\big\},$$

defined by

$$F_{\text{SO}}(e_i)(x) = \begin{cases} \{\mu_i(x)\}, & \mu_i(x) \le \alpha_i \text{ or } \mu_i(x) \ge \beta_i, \\ [\Psi, \Omega], & \alpha_i < \mu_i(x) < \beta_i, \end{cases} \qquad \forall\, x \in U.$$

Here $\{\mu_i(x)\}$ preserves exact under- or over-membership, while $[\Psi, \Omega]$ is the offset-shadowed region *of unresolved soft-membership.*

Example 3.2 (Smartphone Purchase with Offset and Shadow). *Let*

$$U = \{\textit{iPhone}, \textit{Galaxy}, \textit{Pixel}\}, \quad E = \{\textit{Price}, \textit{Battery}\}, \quad \Psi = -0.2,\ \Omega = 1.2.$$

Define offset-soft-memberships:

$$\mu_{\textit{Price}}(\textit{iPhone}) = 0.1,\ \mu_{\textit{Price}}(\textit{Galaxy}) = 0.6,\ \mu_{\textit{Price}}(\textit{Pixel}) = 1.1,$$

$$\mu_{\textit{Battery}}(\textit{iPhone}) = 0.9,\ \mu_{\textit{Battery}}(\textit{Galaxy}) = 0.4,\ \mu_{\textit{Battery}}(\textit{Pixel}) = -0.1.$$

Choose shadow thresholds $\alpha_{\textit{Price}} = 0.3,\ \beta_{\textit{Price}} = 0.8,\ \alpha_{\textit{Battery}} = 0.2,\ \beta_{\textit{Battery}} = 0.7$. Then for Price:

$$F_{\text{SO}}(\textit{Price}) = \big\{(\textit{iPhone}, \{0.1\}),\ (\textit{Galaxy}, [-0.2, 1.2]),\ (\textit{Pixel}, \{1.1\})\big\},$$

and for Battery:

$$F_{\text{SO}}(\textit{Battery}) = \big\{(\textit{iPhone}, \{0.9\}),\ (\textit{Galaxy}, [-0.2, 1.2]),\ (\textit{Pixel}, \{-0.1\})\big\}.$$

Thus "Galaxy" is placed in the offset-shadowed region for both criteria, while "iPhone" and "Pixel" retain precise under/over confidences.

Example 3.3 (Job Candidate Evaluation)**.** *Job Candidate Evaluation is the assessment of applicants' skills, qualifications, and suitability for a job role using various criteria and methods (cf.[73, 74]). Let*

$$U = \{\textit{Ayano}, \textit{Yasuha}, \textit{Carol}\}, \quad E = \{\textit{Experience}, \textit{Interview}\}, \quad \Psi = -0.1,\ \Omega = 1.1.$$

Define offset-soft-memberships

$$\mu_{\textit{Experience}}(\textit{Ayano}) = 0.8,\ \mu_{\textit{Experience}}(\textit{Yasuha}) = 0.5,\ \mu_{\textit{Experience}}(\textit{Carol}) = 0.2,$$

$$\mu_{\textit{Interview}}(\textit{Ayano}) = 0.6,\ \mu_{\textit{Interview}}(\textit{Yasuha}) = 0.9,\ \mu_{\textit{Interview}}(\textit{Carol}) = 0.3.$$

Choose shadow thresholds

$$\alpha_{\textit{Experience}} = 0.4,\ \beta_{\textit{Experience}} = 0.7, \quad \alpha_{\textit{Interview}} = 0.5,\ \beta_{\textit{Interview}} = 0.85.$$

Then the Shadowed Soft Offset F_{SO} gives:

$$F_{\mathrm{SO}}(\textit{Experience}) = \big\{(\textit{Ayano}, \{0.8\}),\ (\textit{Yasuha}, [\Psi, \Omega]),\ (\textit{Carol}, \{0.2\})\big\},$$
$$F_{\mathrm{SO}}(\textit{Interview}) = \big\{(\textit{Ayano}, [\Psi, \Omega]),\ (\textit{Yasuha}, \{0.9\}),\ (\textit{Carol}, \{0.3\})\big\}.$$

Here:

- *Ayano's experience (0.8) exceeds $\beta_{\textit{Experience}}$, so she is classified with exact over-membership $\{0.8\}$.*
- *Yasuha's experience (0.5) lies between $\alpha_{\textit{Experience}}$ and $\beta_{\textit{Experience}}$, so his experience evaluation falls into the offset-shadowed region $[\Psi, \Omega]$.*
- *Carol's experience (0.2) is below $\alpha_{\textit{Experience}}$, yielding exact under-membership $\{0.2\}$.*
- *Dually, Yasuha excels in interview (0.9 $\geq \beta_{\textit{Interview}}$), Carol under-performs (0.3 $\leq \alpha_{\textit{Interview}}$), and Ayano's interview score (0.6) is uncertain.*

Theorem 3.4 (Generalization of Soft Set)**.** *Assume $\Psi = 0, \Omega = 1$, and that each soft-offset membership $\mu_i(x)$ takes only the values 0 or 1. Moreover, suppose the shadow thresholds satisfy $\alpha_i < 0 < 1 < \beta_i$. Then for every $e_i \in E$ and $x \in U$,*

$$F_{\mathrm{SO}}(e_i)(x) = \begin{cases} \{0\}, & \mu_i(x) = 0, \\ \{1\}, & \mu_i(x) = 1, \end{cases}$$

and the "shadowed" case never occurs. Consequently (F_{SO}, E) is identical to the original crisp soft set $(\mathcal{F}, E)$.

Proof. Under the hypothesis, $\mu_i(x) \in \{0, 1\}$. Since $\alpha_i < 0$, the inequality $\mu_i(x) \leq \alpha_i$ holds exactly when $\mu_i(x) = 0$. Likewise, since $\beta_i > 1$, the inequality $\mu_i(x) \geq \beta_i$ holds exactly when $\mu_i(x) = 1$. There is no x for which $\alpha_i < \mu_i(x) < \beta_i$. Hence by definition

$$F_{\mathrm{SO}}(e_i)(x) = \begin{cases} \{\mu_i(x)\} = \{0\}, & \mu_i(x) = 0, \\ \{\mu_i(x)\} = \{1\}, & \mu_i(x) = 1, \end{cases}$$

which coincides with the assignment $\mathcal{F}(e_i) = \{\, x : \mu_i(x) = 1\}$ of the crisp soft set. □

Theorem 3.5 (Generalization of Shadow Soft Set)**.** *If $\Psi = 0$ and $\Omega = 1$, then the Shadowed Soft Offset construction recovers the standard Shadow Soft Set $(F, E)_{\mathcal{SHDW}}$ with shadow-region $[0, 1]$.*

Proof. Setting $[\Psi, \Omega] = [0, 1]$ in the definition yields

$$F_{\mathrm{SO}}(e_i)(x) = \begin{cases} \{\mu_i(x)\}, & \mu_i(x) \le \alpha_i \text{ or } \mu_i(x) \ge \beta_i, \\ [0, 1], & \alpha_i < \mu_i(x) < \beta_i. \end{cases}$$

Since $\mu_i(x) \in [0, 1]$, the singletons $\{\mu_i(x)\}$ reduce to $\{0\}$ or $\{1\}$ precisely when $\mu_i(x) \le \alpha_i$ or $\mu_i(x) \ge \beta_i$. Thus F_{SO} matches exactly the piecewise definition of the Shadow Soft Set $(F, E)_{\mathcal{SHDW}}$. ☐

Theorem 3.6 (Generalization of Shadowed Offset). *When $|E| = 1$ (a single parameter e), the Shadowed Soft Offset reduces to the Shadowed Offset mapping on the universe U.*

Proof. With $E = \{e\}$, the soft-set structure collapses to a single membership function $\mu_e : X \to [\Psi, \Omega]$. The definition of $F_{\mathrm{SO}}(e)$ then coincides term-by-term with the definition of the Shadowed Offset $\mathrm{SO}_{\alpha,\beta}(\tilde{A})$ for the fuzzy offset $\tilde{A}$ having membership μ_e. Hence the two constructions agree. ☐

Theorem 3.7 (Partial Recovery). *Let F_{SO} be the Shadowed Soft Offset defined by $\{\mu_i : U \to [\Psi, \Omega]\}$ and thresholds $\{\alpha_i, \beta_i\}$. Then for each $e_i \in E$ and $x \in U$:*

$$\mu_i(x) \le \alpha_i \implies F_{\mathrm{SO}}(e_i)(x) = \{\mu_i(x)\},$$

$$\mu_i(x) \ge \beta_i \implies F_{\mathrm{SO}}(e_i)(x) = \{\mu_i(x)\}.$$

In other words, whenever $\mu_i(x)$ lies outside the "shadow" band (α_i, β_i), it is exactly recoverable from the singleton output.

Proof. Fix $e_i \in E$ and $x \in U$. By the definition, there are two cases:

- If $\mu_i(x) \le \alpha_i$ or $\mu_i(x) \ge \beta_i$, then

  $$F_{\mathrm{SO}}(e_i)(x) = \{\mu_i(x)\},$$

  so the unique element of the singleton is exactly $\mu_i(x)$.

- If $\alpha_i < \mu_i(x) < \beta_i$, then

  $$F_{\mathrm{SO}}(e_i)(x) = [\Psi, \Omega],$$

  and no unique recovery is possible—but outside this band recovery is exact.

This establishes the claimed partial recovery property. ☐

Theorem 3.8 (Monotonicity). *Let $\{\mu_i\}$ and $\{\nu_i\}$ be two collections of fuzzy-offset soft-memberships with the same thresholds, satisfying $\mu_i(x) \le \nu_i(x)$ for every $e_i \in E$ and $x \in U$. Then*

$$F^{\mu}_{\mathrm{SO}}(e_i)(x) \subseteq F^{\nu}_{\mathrm{SO}}(e_i)(x) \quad \forall\, e_i \in E,\ x \in U,$$

where F^{μ}_{SO} and F^{ν}_{SO} denote the Shadowed Soft Offsets of $\{\mu_i\}$ and $\{\nu_i\}$, respectively.

Proof. Fix e_i and x. Three sub-cases arise:

1. If $\nu_i(x) \le \alpha_i$, then $\mu_i(x) \le \alpha_i$ as well, so

   $$F^{\mu}_{\mathrm{SO}}(e_i)(x) = \{\mu_i(x)\}, \quad F^{\nu}_{\mathrm{SO}}(e_i)(x) = \{\nu_i(x)\},$$

   and $\mu_i(x) \le \nu_i(x)$ implies $\{\mu_i(x)\} \subseteq \{\nu_i(x)\}$.

2. If $\mu_i(x) \geq \beta_i$, then $\nu_i(x) \geq \beta_i$ also, and
$$F^{\mu}_{\mathrm{SO}}(e_i)(x) = \{\mu_i(x)\}, \quad F^{\nu}_{\mathrm{SO}}(e_i)(x) = \{\nu_i(x)\},$$
with $\mu_i(x) \leq \nu_i(x)$ giving the required inclusion.

3. Otherwise $\alpha_i < \mu_i(x) \leq \nu_i(x) < \beta_i$. Then
$$F^{\mu}_{\mathrm{SO}}(e_i)(x) = F^{\nu}_{\mathrm{SO}}(e_i)(x) = [\Psi, \Omega],$$
so trivially $[\Psi, \Omega] \subseteq [\Psi, \Omega]$.

In all cases $F^{\mu}_{\mathrm{SO}}(e_i)(x) \subseteq F^{\nu}_{\mathrm{SO}}(e_i)(x)$, proving monotonicity. □

4. Conclusion and future works

In this paper, we introduced the novel concept of the *Shadowed Offset* and presented a concise study of its fundamental properties. As part of future work, we aim to explore extensions of this concept using Neutrosophic Sets [75] and Quadri-Partitioned Neutrosophic Sets [76]. We also intend to investigate its applicability to graph-theoretic frameworks by incorporating structures such as Hypergraphs [77–79] and SuperHypergraphs [80–86].

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

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