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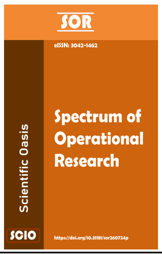

Fuzzy Logic Base Large-Scale Urban Traffic Signal Optimization Using Circular Complex Pythagorean Fuzzy Sets and Intelligent Hybrid Machine Learning

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ABSTRACT

Large-scale urban traffic networks operate under highly dynamic, nonlinear, and uncertain conditions, where fluctuating traffic demand, incomplete sensor data, and unpredictable driver behavior make real-time signal optimization a complex decision-making challenge. To address these issues, this study introduces the concept of Circular Complex Pythagorean Fuzzy Sets (CrC-PFS) for modeling uncertainty in urban traffic signal control problems. The proposed CrC-PFS framework extends classical Pythagorean fuzzy sets, Complex Pythagorean fuzzy sets, and circular Pythagorean fuzzy environments by providing greater flexibility in representing membership, non-membership, and hesitation degrees under periodic and time-varying traffic patterns. To enhance computational capability, we establish refined algebraic operational laws for CrC-PFS, including direct sum, direct product, and scalar multiplication operators based on generalized t-norm and t-conorm structures. In addition, Circular Complex Pythagorean fuzzy weighted averaging and ordered weighted aggregation operators are developed to integrate multiple traffic performance indicators such as queue length, delay time, saturation flow, and emission levels within a multi-criteria decision-making framework. Furthermore, an intelligent hybrid machine learning mechanism is incorporated to dynamically learn traffic flow patterns and adapt signal timing strategies in large-scale urban networks. By integrating fuzzy uncertainty modeling with predictive learning algorithms, a robust optimization framework is constructed for adaptive traffic signal coordination. The experimental findings demonstrate that the proposed model significantly improves traffic efficiency, reduces congestion levels, and enhances overall network resilience, thereby supporting sustainable and uncertainty-aware urban transportation management.

Keywords:
Circular Complex Pythagorean fuzzy sets fuzzy set; Complex Pythagorean fuzzy sets; Multi-criteria decision-making; MCDM

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1. Introduction

Large-scale urban traffic management represents a critical and challenging decision-making problem in modern smart cities, where traffic signal control systems must continuously determine optimal signal timings, phase sequences, and coordination strategies from multiple feasible alternatives. These decisions are governed by numerous and often conflicting criteria, including traffic flow efficiency, vehicle delay, queue length, pedestrian safety, fuel consumption, and environmental emissions. With the rapid growth of urban populations and vehicle density, traditional fixed-time or rule-based signal control methods are increasingly inadequate for handling complex and dynamic traffic conditions.

Urban traffic environments are inherently uncertain and nonlinear due to fluctuating traffic demand, unpredictable driver behavior, weather variations, road incidents, and sensor noise. Conventional crisp or probabilistic decision-making models struggle to effectively represent such ambiguity and hesitation in traffic data. To overcome these limitations, fuzzy set–based decision frameworks have gained significant attention for their ability to model imprecision, partial truth, and uncertainty. In particular, *circular complex Pythagorean fuzzy sets* provide a powerful mathematical structure to capture both the magnitude and phase information of traffic-related uncertainty, enabling a richer and more flexible representation of complex traffic states.

In parallel, intelligent hybrid machine learning techniques have demonstrated strong capabilities in learning traffic patterns, predicting congestion, and adapting control strategies in real time. However, machine learning models alone often lack interpretability and robustness when faced with incomplete or noisy data. By integrating circular complex Pythagorean fuzzy modeling with intelligent hybrid machine learning, urban traffic signal optimization systems can effectively fuse uncertain traffic information, extract meaningful patterns, and support reliable multi-attribute decision-making.

Such an intelligent decision framework enables adaptive, scalable, and uncertainty-aware traffic signal control across large urban networks. The resulting optimization improves traffic efficiency, reduces congestion and emissions, enhances road safety, and contributes to sustainable urban mobility, thereby supporting the development of resilient and intelligent transportation systems in smart cities.

1.1 *Literature Review*

The concept of the fuzzy set was introduced by Zadeh in 1965 [1] to model real-world problems involving uncertainty and imprecise information. Fuzzy set theory allows elements to belong to a set with varying degrees of membership and has been widely applied in areas such as data mining, clustering, decision analysis, and medical research.

Later, Atanassov [2] extended this concept by introducing intuitionistic fuzzy sets, which consider both membership and non-membership degrees. However, the restrictions of this model limit its ability to represent some real-world situations. To overcome this issue, Yager [3] proposed the Pythagorean fuzzy set, providing greater flexibility. Subsequently, Senapati [5] introduced Fermatean fuzzy sets to further expand the representation of uncertainty. Yager later generalized these concepts through the q-rung orthopair fuzzy set [4], offering a more flexible framework for handling uncertain information.

In many real-world decision-making problems, experts may express opinions as positive, negative, or neutral. To capture these responses, Cuong [6] proposed picture fuzzy sets, which include membership, non-membership, and neutrality degrees. Later, Mahmood et al. [7] introduced spherical fuzzy sets, providing a more flexible framework than earlier models. To further generalize this concept, Ullah et al. [8] proposed T-spherical fuzzy sets, which offer enhanced capability for modeling complex uncertainty.

These developments in fuzzy set theory provide powerful tools for representing uncertainty and have been widely applied in intelligent decision-making, sustainability analysis, and various technological applications.

In real-life decision-making scenarios, decision makers (DMs) often face significant challenges when selecting the most appropriate option among several alternatives, especially as systems become more complex. Organizations must consider multiple objectives simultaneously while shaping strategies, defining goals, and motivating their workforce. Therefore, effective decision-making requires methods that can properly handle uncertainty and provide reliable evaluations of available alternatives. However, many existing approaches have limitations in representing uncertain information and its variations over time.

To address this issue, [9] introduced the concept of complex fuzzy sets (CFS), which extend the traditional fuzzy set framework by allowing membership values to be represented in the complex plane. This extension enables the modeling of additional information such as amplitude and phase terms. Since CFS considers only complex-valued membership information, [10] later proposed complex intuitionistic fuzzy sets (CIFS) by incorporating a complex-valued non-membership function. Although CIFS improved the representation of uncertainty, its restrictions still limit its ability to describe certain complex situations.

To provide greater flexibility, Ullah et al. [11] developed the complex Pythagorean fuzzy set (CPyFS), which extends CIFS by relaxing its constraints and allowing a broader range of membership and non-membership information. However, some decision-making situations may still violate the conditions of these models. To overcome these limitations, Liu et al. [13] proposed the complex q-rung orthopair fuzzy set (Cq-ROFS), which provides a more generalized framework for handling uncertain and complex data in multi-attribute decision-making (MADM) problems.

Although Cq-ROFS improves the representation of uncertainty, it does not explicitly consider neutral information. To capture such situations, Akram et al. [14] introduced the complex picture fuzzy set (CPiFS), which incorporates membership, non-membership, and neutral information simultaneously. Later, the complex spherical fuzzy set (CSFS) [15] was proposed as a further generalization of CPiFS, providing greater flexibility in representing uncertain data. More recently, Ali et al. [16] introduced the complex T-spherical fuzzy set (CT-SFS), which offers an even more generalized framework capable of handling a wider range of complex and uncertain decision-making situations..

Another research direction investigated the possibility of extending the framework of intuitionistic fuzzy sets to a circular domain. In this context, Atanassov [17] proposed the concept of Circular Intuitionistic Fuzzy Sets (CIFSs), where membership and non-membership information are represented within a circular region. This representation provides additional flexibility for modeling uncertain information.

Later, Bozyigit et al. [18] introduced Circular Pythagorean Fuzzy Sets (Cr-PFSs) as a further extension of circular intuitionistic models. In this approach, membership and non-membership values are represented within a circular structure characterized by a radius that determines the allowable region on the plane. This formulation enables a broader representation of uncertainty compared with earlier fuzzy models.

More recently, Yusoff et al. [19] generalized this idea by proposing Circular q-rung Orthopair Fuzzy Sets (Cirq-ROFSs), which extend the circular fuzzy framework by introducing a parameter that increases the flexibility of membership and non-membership representation. Zeeshan et al. [21] further investigated the mathematical properties of Cirq-ROFSs and developed several operational laws, including algebraic and Dombi operators.

In recent years, fuzzy multi-criteria decision-making (MCDM) approaches have been widely applied in business, engineering, and sustainability-related problems. For example, Ullah et al. [29] proposed a stock market decision-making framework based on circular complex picture fuzzy sets combined with the CRITIC–WASPAS method. Liu et al. [31] introduced a prospect–regret based three-way decision model using q-rung orthopair fuzzy preference relations to address energy crisis issues. In the field of sustainable development, Liu et al. [32] applied CODAS and WASPAS methods under circular linguistic T-spherical fuzzy Hamy mean aggregation operators for green building evaluation. Furthermore, Ali et al. [33] investigated artificial intelligence applications in healthcare using a CRITIC–TOPSIS approach based on λ-(pq) cubic quasi q-rung orthopair fuzzy aggregation operators. Recent research has also introduced advanced circular and complex fuzzy frameworks to address uncertainty in intelligent decision-making systems, particularly in applications such as agricultural robotics [30].

1.2 *The main motivations*

Overall, circular complex pythagorean fuzzy set (CrC-PFS) is the extensive generalization of Pythagorean fuzzy set. Nonetheless, some situations cannot be managed well by C-PFS. The proposed within the framework of this paper is the new method of Circular Complex pythagorean fuzzy set Fuzzy Set (CrC-PFS) whose operational laws include CrC-PFS a combination of the degrees of membership, abstinence, and non-membership with a condition in which the total of power of the real part (as well as imaginary part) of the membership, abstinence, and non-membership grades is not more than a unit interval.

MCDM methods are progressively gaining popularity as future devices of evaluating and solving complex real-time dilemmas owing to their inherent ability to evaluate a large number of choices by a number of variables with the aim of possibly selecting the most suitable choice. The challenges in MCDM have several distinctiveness like the existence of more than one non-commensurable and conflicting criteria, the criteria have different units of measurement, and the existence of relatively dissimilar options, as well. These decision-making issues characterize multidimensional situations and are being addressed using the MCDM techniques. The MCDM techniques are aimed mostly at the analysis and prioritization of the available alternatives. This is so many times because MCDM techniques can give different results (i.e. the same alternatives are ranked differently depending on which techniques are applied). This is due to the numerous mathematical artifacts in which the methodologies under discussion make use of them. The WASPAS technique is a peculiar blend of two popular MCDM techniques i.e. weighted sum model (WSM) and weighted product model (WPM). Its use involves first developing decision/evaluation matrix, $X = [x_{ij}]_{mxn}$ where the x_{ij} is the performance of i_{th} alternative against j_{th} criterion where m and n are the number of alternatives and number of criteria respectively. The multiple options that are now feasible are ranked in terms of the value of Q and the top alternative is that with the maximum value of Q. Table 1 presents the exact limitations of the previous studies against the suggested strategy.

Table 1
Comparison of the proposed model with extant models in the literature

Concept	**MD**	**NMD**	**CMD**	**CNMD**	**Radius**	λ	β
FS	Yes	No	No	No	No	No	No
IFS	Yes	Yes	No	No	No	No	No
PyFS	Yes	Yes	No	No	No	No	No
q-ROFS	Yes	Yes	No	No	No	No	No
PiFS	Yes	Yes	No	No	No	No	No
SFS	Yes	Yes	No	No	No	No	No
T-SFS	Yes	Yes	No	No	No	No	No
CrIFS	Yes	Yes	No	No	Yes	No	No
CrPyFS	Yes	Yes	No	No	Yes	No	No
Crq-ROFS	Yes	Yes	No	No	Yes	No	No
CFS	Yes	No	Yes	No	No	No	No
CIFS	Yes	Yes	Yes	Yes	No	No	No
CPyFS	Yes	Yes	Yes	Yes	No	No	No
Cq-ROFS	Yes	Yes	Yes	Yes	No	No	No
CPiFS	Yes	Yes	Yes	Yes	No	No	No
CSFS	Yes	Yes	Yes	Yes	No	No	No
CT-SFS	Yes	Yes	Yes	Yes	No	No	No
Proposed CrC-PFS	Yes	Yes	Yes	Yes	Yes	Yes	Yes

where MD is membership degree, NMD is non membership degree, CMD is complex membership degree, CNMD is complex non membership degree, λ is acts as a power-based aggregation parameter, and β is acts as a control coefficient within the WASPAS method.

We found from the aforementioned investigation that the following are the main issues that all experts have:

(i) On the basis of CrC-PFSs, how should novel operational laws be drafted?

(ii) On the basis of novel operational laws, how may new operators be developed?

(iii) How can all actions be ranked according to the developed operators?

1.3 *Novelty and main contributions*

This research is intended to establish a rational and intellectual method of supporting a decision so that the most appropriate alternative can be adopted out of a number of alternative sources. The integration of algebraic complex operational rules into the CrC-PFS environment will allow one to use the CrC-PFS arithmetic and geometric mean aggregation operators which will ensure the efficiency of the conceptual framework.

The key achievements and objectives of this article are as follows:

1. It is more competent, comprehensive and reliable than the present day conception such as CrC-PFS in the aspect of uncertain data to be dealt with in the decision making process. Also, no prior studies have examined the relationship between CrC-PFS situations and the use of arithmetic

and geometric mean aggregation operators based on algebraic complex operations laws. Thus, it is important to improve geometrical and arithmetic mean of aggregation operators based on complex operational principles to solve MCDM problems in CrC-PFS cases.

2. An interesting concept to managing data in three dimensions in one set is the arithmetic and geometric mean aggregation operators pursuant to setting CrC-PFS that uses complex operational rules. Thus, this research aims at providing Crcular Complex-PFS weighted arithmetic mean aggregation operator (CrC-PFWAM), Crcular Complex FFS ordered weighted arithmetic mean aggregation operator (CrC-PSFOWAM), Crcular Complex PiSFS weighted geometric mean aggregation operator (CrC-PFWGM) and Crcular Complex FFS ordered weighted geometric mean aggregation operator (CrC-PFOWGM).

3. The relations defining between these operators are emphasized to deal with some of their features such as being bounded, idempent and monotonic.

4. To develop two different, innovative approaches founded on the CrC-PFWAM and CrC-PFWGM.

5. An illustrative representation of the given approach is provided to make the proposed procedure even more understandable and clear. The recommended modelling technique is graphically presented by applying a flowchart in visualising the desired process.

6. To give examples of application that can prove the feasibility and reliability of the proposed techniques. Also, the comparison of the proposed techniques with the existing methods will show that the proposed techniques are superior and that the aggregation process will be more flexible as the arithmetic and geometric mean aggregation operators are used in accordance with CrC-PFS conditions and the complicated rules of operation.

1.4 *The structure of this paper*

As shown below, this article is structured as follows: Section 2 deals with the construct of complex operational laws based CrC-PFWAM aggregation operator, CrC-PFOWAM aggregation operator (CrC-PFOWAM). Section 3 discusses the concept of CrC-PFWGM aggregation operator and CrC-PFOWGM aggregation operator in the framework of Circular Complex pythagorean fuzzy sets and their properties. Section 4 proposes an innovative method of decision-making through the latest methods that rely on the CrC-PFWAM and CrC-PFWGM. Besides, Section 5 includes an example to illustrate the worth of the proposed strategy to choose. Section 6 defines the comparability and sensitivity analysis which show the logic and stability of the proposed technique. Section 7 gives a conclusion to the article.

2. Circular Complex pythagorean fuzzy weighted arithmetic mean aggregation operators

We provide definitions of Circular Complex pythagorean fuzzy operational laws for CrC-PFNs in the following section. Following them, several aggregation (circular complex pythagorean fuzzy weighted arithmetic mean aggregation operator (CrC-PFWAM), circular complex PiSF ordered weighted arithmetic mean aggregation operator (CrC-PFOWAM)) operators based on circular complex pythagorean fuzzy operational laws will be created.

Definition 1. *[9] A CFS A is defined as:*

$$A = \{(x, \eta(x)) \mid x \in X\}$$

where $\eta(x) = \eta(x)e^{i2\pi\omega(x)}$ *denotes the grade of complex-valued truth with a condition:* $0 \leq \eta(x), \omega(x) \leq 1$.

Definition 2. *[12] Let* X *be a universal set. A complex pythagorean fuzzy set (CFFS)* $\widetilde{A}$ *in* X *is defined as*

$$\widetilde{A} = \left\{ \left(x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x)\right) : x \in X \right\},$$

where $\mu_{\widetilde{A}}(x)$ *and* $\nu_{\widetilde{A}}(x)$ *denote the complex-valued membership and non-membership degrees, respectively, and are given by*

$$\mu_{\widetilde{A}}(x) = \mu_r(x)e^{i\theta_\mu(x)}, \qquad \nu_{\widetilde{A}}(x) = \nu_r(x)e^{i\theta_\nu(x)},$$

with

$$0 \leq \mu_r(x) \leq 1, \qquad 0 \leq \nu_r(x) \leq 1, \qquad \mu_r^2(x) + \nu_r^2(x) \leq 1.$$

The hesitation (indeterminacy) degree corresponding to x *is defined as*

$$\pi_{\widetilde{A}}(x) = \sqrt{1 - \mu_r^2(x) - \nu_r^2(x)}\, e^{i\theta_\pi(x)},$$

where $\theta_\mu(x), \theta_\nu(x), \theta_\pi(x) \in [0, 2\pi)$.

2.1 *Proposed Circular Complex Pythagorean fuzzy Sets*

One aim of this study is to explore the novel approach of CrC-PFSs and their operational laws. These operational laws are also verified with the help of a numerical example.

Definition 3. *A* CrC*-PFSP is defined as:*

$$P = \{(x, \eta(x), \phi(x), r(x)) \mid x \in X\}$$

where $\eta(x) = \eta_{C_1}e^{i2\pi\eta_{C_1}^{+im}}$, $\phi(x) = \phi_{C_1}e^{i2\pi\phi_{C_1}^{+im}}$, *and* $r(x) = r_{C_1}e^{i2\pi r_{C_1}^{+im}}$ *denote the membership degree, non-membership and radius with the conditions:* $0 \leq \eta_{C_1}^2 + \phi_{C_1}^2 \leq 1$ *and* $0 \leq (\eta_{C_1}^{i2} + \phi_{C_1}^{i2}) \leq 1$*. Additionally, the term* $H(x) = Re^{i2\pi\omega_R(x)}$ *such that* $R = \left(1 - \eta_{C_1}^2 + \phi_{C_1}^2\right)^{1/2}$ *and* $\omega_R(x) = \left(1 - (\eta_{C_1}^{i2} + \phi_{C_1}^{i2})\right)^{1/2}$ *expresses the complex hesitancy grade of* x*. Moreover,*
$P = \left(\eta_{C_1}e^{i2\pi\eta_{C_1}^{+im}}, \phi_{C_1}e^{i2\pi\phi_{C_1}^{+im}}, r_{C_1}e^{i2\pi r_{C_1}^{+im}}\right)$ *is called a* CrC*-PFN.*

Definition 4. *For any* CrC*-PFN.* $P_1 = \left(\eta_{C_1}e^{i2\pi\eta_{C_1}^{+im}}, \phi_{C_1}e^{i2\pi\phi_{C_1}^{+im}}, r_{C_1}e^{i2\pi r_{C_1}^{+im}}\right)$*, the score and accuracy functions are defined by*

$$SC\left(P_1\right) = \frac{1}{8}\left\{ \left(\eta_{C_1}\right)^2 + \left(\eta_{C_1}^{im}\right)^2 - \left(\phi_{C_1}\right)^2 - \left(\phi_{C_1}^{im}\right)^2 + \left(r_{C_1}\right)^2 + \left(r_{C_1}^{im}\right)^2 \right\}$$

and

$$AC\left(P_1\right) = \frac{1}{8}\left\{ \left(\eta_{C_1}\right)^2 + \left(\eta_{C_1}^{im}\right)^2 + \left(\phi_{C_1}\right)^2 + \left(\phi_{C_1}^{im}\right)^2 + \left(r_{C_1}\right)^2 + \left(r_{C_1}^{im}\right)^2 \right\}$$

where $SC\left(P_1\right) \in [-1, 1]$ *and* $AC\left(P_1\right) \in [0, 1]$.

Definition 5. *Let* $P_1 = \left(\eta_{C_1}e^{i2\pi\eta_{C_1}^{+im}}, \phi_{C_1}e^{i2\pi\phi_{C_1}^{+im}}, r_{C_1}e^{i2\pi r_{C_1}^{+im},}\right)$,
$P_2 = \left(\eta_{C_2}e^{i2\pi\eta_{C_1}^{+im}}, \phi_{C_2}e^{i2\pi\phi_{C_2}^{+im}}, r_{C_1}e^{i2\pi r_{C_2}^{+im},}\right)$ *be two CrC-PFNs. Then*
(1) *if* $SC\left(P_1\right) > SC\left(P_2\right)$, *then* $P_1 > P_2$,
(2) *if* $SC\left(P_1\right) = SC\left(P_2\right)$ *then*
(i) *if* $AC\left(P_1\right) > AC\left(P_2\right)$, *then* $P_1 > P_2$,
(ii) *if* $AC\left(P_1\right) = AC\left(P_2\right)$, *then* $P_1 = P_2$.

2.2 *Algebraic Circular Complex fermatean fuzzy operational laws*

The new direct sum, direct product, and scalar multiplication operations are defined for CrC-PFSs in this subsection.

Definition 6. *Let* $P_1 = \left(\eta_{C_1} e^{i2\pi\eta_{C_1}^{+im}}, \phi_{C_1} e^{i2\pi\phi_{C_1}^{+im}}, r_{C_1} e^{i2\pi r_{C_1}^{+im},}\right)$,

$P_2 = \left(\eta_{C_2} e^{i2\pi\eta_{C_1}^{+im}}, \phi_{C_2} e^{i2\pi\phi_{C_2}^{+im}}, r_{C_1} e^{i2\pi r_{C_2}^{+im},}\right)$ *be two* CrC *-PFNs. Then we define the following algebraic Circular Complex fermatean fuzzy operational laws:*

$$\text{(i) } P_1 \oplus^1 P_2 = \left(\begin{array}{c} \left(\frac{1}{2}\left(\left(\eta_{C_1}\right)^2 + \left(\eta_{C_2}\right)^2\right)\right)^{\frac{1}{2}} .e^{i2\pi\left(\frac{1}{2}\left(\left(\eta_{C_1}^{im}\right)^2+\left(\eta_{C_2}^{im}\right)^2\right)\right)^{\frac{1}{2}}}, \\ \left(1-\frac{1}{2}\left(\left(\phi_{C_1}\right)^2 + \left(\phi_{C_2}\right)^2\right)\right)^{\frac{1}{2}} .e^{i2\pi\left(1-\frac{1}{2}\left(\left(\phi_{C_1}^{+im}\right)^2+\left(\phi_{C_2}^{+im}\right)^2\right)\right)^{\frac{1}{2}}}, \\ \left(\frac{1}{2}\left(\left(r_{C_1}\right)^2 + \left(r_{C_2}\right)^2\right)\right)^{\frac{1}{2}} .e^{i2\pi\left(\frac{1}{2}\left(\left(r_{C_1}^{im}\right)^2+\left(r_{C_2}^{im}\right)^2\right)\right)^{\frac{1}{2}}} \end{array}\right);$$

$$\text{(ii) } P_1 \oplus^2 P_2 = \left(\begin{array}{c} \left(\frac{1}{2}\left(\left(\eta_{C_1}\right)^2 + \left(\eta_{C_2}\right)^2\right)\right)^{\frac{1}{2}} .e^{i2\pi\left(\frac{1}{2}\left(\left(\eta_{C_1}^{im}\right)^2+\left(\eta_{C_2}^{im}\right)^2\right)\right)^{\frac{1}{2}}}, \\ \left(1-\frac{1}{2}\left(\left(\phi_{C_1}\right)^2 + \left(\phi_{C_2}\right)^2\right)\right)^{\frac{1}{2}} e^{i2\pi\left(1-\frac{1}{2}\left(\left(\phi_{C_1}^{+im}\right)^2+\left(\phi_{C_2}^{+im}\right)^2\right)\right)^{\frac{1}{2}}}, \\ \left(1-\frac{1}{2}\left(\left(r_{C_1}\right)^2 + \left(r_{C_2}\right)^2\right)\right)^{\frac{1}{2}} e^{i2\pi\left(1-\frac{1}{2}\left(\left(r_{C_1}^{+im}\right)^2+\left(r_{C_2}^{+im}\right)^2\right)\right)^{\frac{1}{2}}} \end{array}\right)$$

$$\text{(iii) } P_1 \otimes^1 P_2 = \left(\begin{array}{c} \left(1-\frac{1}{2}\left(\left(\eta_{C_1}\right)^2 + \left(\eta_{C_2}\right)^2\right)\right)^{\frac{1}{2}} .e^{i2\pi\left(\frac{1}{2}\left(\left(\eta_{C_1}^{+im}\right)^2+\left(\eta_{C_2}^{+im}\right)^2\right)\right)^{\frac{1}{2}}}, \\ \left[\left(\frac{1}{2}\left(\left(\phi_{C_1}\right)^2 + \left(\phi_{C_2}\right)^2\right)\right)^{\frac{1}{2}}\right] .e^{i2\pi\left(\frac{1}{2}\left(\left(\phi_{C_1}^{+im}\right)^2+\left(\phi_{C_2}^{+im}\right)^2\right)\right)^{\frac{1}{2}}}, \\ \left(1-\frac{1}{2}\left(\left(r_{C_1}\right)^2 + \left(r_{C_2}\right)^2\right)\right)^{\frac{1}{2}} e^{i2\pi\left(\frac{1}{2}\left(\left(r_{C_1}^{+im}\right)^2+\left(r_{C_2}^{+im}\right)^2\right)\right)^{\frac{1}{2}}}, \end{array}\right);$$

$$\text{(iii) } P_1 \otimes^2 P_2 = \left(\begin{array}{c} \left(1-\frac{1}{2}\left(\left(\eta_{C_1}\right)^2 + \left(\eta_{C_2}\right)^2\right)\right)^{\frac{1}{2}} .e^{i2\pi\left(\frac{1}{2}\left(\left(\eta_{C_1}^{+im}\right)^2+\left(\eta_{C_2}^{+im}\right)^2\right)\right)^{\frac{1}{2}}}, \\ \left[\left(\frac{1}{2}\left(\left(\phi_{C_1}\right)^2 + \left(\phi_{C_2}\right)^2\right)\right)^{\frac{1}{2}}\right] .e^{i2\pi\left(\frac{1}{2}\left(\left(\phi_{C_1}^{+im}\right)^2+\left(\phi_{C_2}^{+im}\right)^2\right)\right)^{\frac{1}{2}}}, \\ \left[\left(\frac{1}{2}\left(\left(r_{C_1}\right)^2 + \left(r_{C_2}\right)^2\right)\right)^{\frac{1}{2}}\right] .e^{i2\pi\left(\frac{1}{2}\left(\left(r_{C_1}^{+im}\right)^2+\left(r_{C_2}^{+im}\right)^2\right)\right)^{\frac{1}{2}}} \end{array}\right);$$

$\text{(iii) } \alpha^1 P = \left(\begin{array}{c} (\alpha)^{\frac{1}{2}} \eta_C e^{i2\pi(\alpha)^{\frac{1}{2}}\eta_C^{+im}}, \left(\alpha\left(1-\left(\phi_C\right)^2\right)\right)^{\frac{1}{2}} .e^{i2\pi\left(\alpha\left(1-\left(\phi_C^{+im}\right)^2\right)\right)^{\frac{1}{2}}}, \\ (\alpha)^{\frac{1}{2}} r_C .e^{i2\pi(\alpha)^{\frac{1}{2}} r_C^{+im}} \end{array}\right)$, *where* $0 \leq \alpha \leq 1$;

$\text{(iii) } \alpha^2 P = \left(\begin{array}{c} (\alpha)^{\frac{1}{2}} \eta_C e^{i2\pi(\alpha)^{\frac{1}{2}}\eta_C^{+im}}, \left(\alpha\left(1-\left(\phi_C\right)^2\right)\right)^{\frac{1}{2}} .e^{i2\pi\left(\alpha\left(1-\left(\phi_C^{+im}\right)^2\right)\right)^{\frac{1}{2}}}, \\ \left(\alpha\left(1-\left(r_C\right)^2\right)\right)^{\frac{1}{2}} .e^{i2\pi\left(\alpha\left(1-\left(r_C^{+im}\right)^2\right)\right)^{\frac{1}{2}}} \end{array}\right)$, *where* $0 \leq \alpha \leq 1$;

$\text{(iv) } P^{\lambda} = \left(\begin{array}{c} l^{\frac{1}{2}} \left(\eta_C\right)^{\lambda} .e^{i2\pi r^{\frac{1}{3}}\left(\eta_C^{+im}\right)^{\lambda}}, l^{\frac{1}{2}} \left(\phi_C\right)^{\lambda} .e^{i2\pi r^{\frac{1}{2}}\left(\phi_C^{+im}\right)^{\lambda}} \\ , l^{\frac{1}{2}} \left(r_C\right)^{\lambda} .e^{i2\pi r^{\frac{1}{2}}\left(r_C^{+im}\right)^{\lambda}} \end{array}\right)$, *where* l *is the total number of* CrC*-PFNs that are a part of the procedure*;

$$\text{(v) } P^{\odot 1\alpha} = \left(\begin{array}{c} \left(\alpha\left(1-\left(\eta_C\right)^2\right)\right)^{\frac{1}{3}} .e^{i2\pi\left(\alpha\left(1-\left(\eta_C^{+im}\right)^2\right)\right)^{\frac{1}{2}}}, (\alpha)^{\frac{1}{2}} \phi_C .e^{i2\pi(\alpha)^{\frac{1}{2}}\phi_C^{+im}}, \\ \left(\alpha\left(1-\left(r_C\right)^2\right)\right)^{\frac{1}{2}} .e^{i2\pi\left(\alpha\left(1-\left(r_C^{+im}\right)^2\right)\right)^{\frac{1}{2}}} \end{array}\right),$$

where $0 \leq \alpha \leq 1$.

$$\text{(vi) } P^{\odot 2\alpha} = \left(\begin{array}{c} \left(\alpha\left(1-\left(\eta_C\right)^2\right)\right)^{\frac{1}{3}} .e^{i2\pi\left(\alpha\left(1-\left(\eta_C^{+im}\right)^2\right)\right)^{\frac{1}{2}}}, (\alpha)^{\frac{1}{2}} \phi_C .e^{i2\pi(\alpha)^{\frac{1}{2}}\phi_C^{+im}}, \\ (\alpha)^{\frac{1}{2}} r_C .e^{i2\pi(\alpha)^{\frac{1}{2}} r_C^{+im}} \end{array}\right),$$

where $0 \leq \alpha \leq 1$.

Definition 7. *Let* $\left\{P_j = \left(\eta_{C_j} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_j} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_j} e^{i2\pi r^{+im}_{C_j},}\right) : j = 1, 2, ..., m\right\}$ *be the collection of* C_rC*-PF values and let* C_rC*-*$PFWAM : \Omega^m \to \Omega$*. If*

C_rC*-*$PFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\left(\alpha^1_1 P_1\right)^\lambda \oplus \left(\alpha^1_2 P_2\right)^\lambda \oplus \left(\alpha^1_3 P_3\right)^\lambda \oplus ... \oplus \left(\alpha^1_m P_m\right)^\lambda\right)^{\frac{1}{\lambda}}$ *then* C_rC*-*$PFWAM$ *is called a Circular Complex fermatean fuzzy weighted averaging mean operator of dimension* n *, where* Ω *is the set of all* C_rC*-PF values,* $E = \left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0, 1]$ *,* $\Sigma^m_{r=1}\alpha_r = 1$*, where* $r = 1, 2, ..., m$*.*

Theorem 1. *Let* $\left\{P_j = \left(\eta_{C_j} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_j} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_j} e^{i2\pi r^{+im}_{C_j},}\right) : j = 1, 2, ..., m\right\}$ *be the collection of* C_rC*-PF values. Then by using the* C_rC*-*$PFWAM_E$ *operator their aggregated value is also a* C_rC*-PF value and*

$$\begin{aligned} & C^1_r C - PFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \\ = & \left(\begin{array}{c} \left(\left(\sum_{r=1}^m \left(\alpha_j \left(\eta_{C_j}\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left(\left(\sum\limits_{j=1}^m \left(\alpha_r \left(\eta^{+im}_{C_j}\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}}, \\ \left(\left(1 - \sum\limits_{j=1}^m \left(\alpha_j \left(1 - \left(\phi_{C_j}\right)^2\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left(\left(1 - \sum\limits_{j=1}^m \left(\alpha_j \left(1 - \left(\phi^{+im}_{C_j}\right)^2\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}}, \\ \left(\left(\sum\limits_{j=1}^m \left(\alpha_j \left(r_{C_j}\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left(\left(\sum\limits_{j=1}^m \left(\alpha_j \left(r^{+im}_{C_j}\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}} \end{array}\right). \end{aligned}$$

$E = \left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_n\right)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0, 1]$ *and* $\Sigma^n_{r=1}\alpha_r = 1, r = 1, 2, ..., m$*.*

Proof. Let

$$\alpha^1_1 P_1 = \left(\begin{array}{c} \left(\alpha_1\right)^{\frac{1}{2}} \eta_{C_1}.e^{i2\pi\left(\alpha_1\right)^{\frac{1}{2}}\eta^{+im}_{C_1}}, \left(\alpha_1\left(1 - \left(\phi_{C_1}\right)^2\right)\right)^{\frac{1}{2}} .e^{i2\pi\left(\alpha_1\left(1 - \left(\phi^{+im}_{C_1}\right)^2\right)\right)^{\frac{1}{2}}}, \\ \left(\alpha_1\right)^{\frac{1}{2}} r_{C_1}.e^{i2\pi\left(\alpha_1\right)^{\frac{1}{2}} r^{+im}_{C_1}} \end{array}\right)$$

and

$$\alpha^1_2 P_2 = \left(\begin{array}{c} \left(\alpha_2\right)^{\frac{1}{2}} \eta_{C_2} e^{i2\pi\left(\alpha_2\right)^{\frac{1}{2}}\eta^{+im}_{C_2}}, \left(\alpha_2\left(1 - \left(\phi_{C_2}\right)^2\right)\right)^{\frac{1}{2}} e^{i2\pi\left(\alpha_2\left(1 - \left(\phi^{+im}_{C_2}\right)^2\right)\right)^{\frac{1}{2}}}, \\ \left(\alpha_2\right)^{\frac{1}{2}} r_{C_2} e^{i2\pi\left(\alpha_2\right)^{\frac{1}{2}} r^{+im}_{C_2}} \end{array}\right)$$

Then

$$\left(\alpha^1_1 P_1\right)^\lambda = \left(\begin{array}{c} l^{\frac{1}{2}}\left(\left(\alpha_1\right)^{\frac{1}{2}} \eta_{C_1}\right)^\lambda e^{i2\pi l^{\frac{1}{2}}\left(\left(\alpha_1\right)^{\frac{1}{2}}\eta^{+im}_{C_1}\right)^\lambda}, \\ l^{\frac{1}{2}}\left(\left(\alpha_1\left(1 - \left(\phi_{C_1}\right)^2\right)\right)^{\frac{1}{2}}\right)^\lambda e^{i2\pi l^{\frac{1}{2}}\left(\left(\alpha_1\left(1 - \left(\phi^{+im}_{C_1}\right)^2\right)\right)^{\frac{1}{2}}\right)^\lambda}, \\ l^{\frac{1}{2}}\left(\left(\alpha_1\right)^{\frac{1}{2}} r_{C_1}\right)^\lambda e^{i2\pi l^{\frac{1}{2}}\left(\left(\alpha_1\right)^{\frac{1}{2}} r^{+im}_{C_1}\right)^\lambda} \end{array}\right)$$

and

$$\left(\alpha_2^1 P_2\right)^{\lambda}=\begin{pmatrix} l^{\frac{1}{2}}\left(\left(\alpha_2\right)^{\frac{1}{2}}\eta_{C_2}\right)^{\lambda}e^{i2\pi l^{\frac{1}{2}}\left(\left(\alpha_2\right)^{\frac{1}{2}}\eta_{C_2}^{+im}\right)^{\lambda}}, \\ l^{\frac{1}{2}}\left(\left(\alpha_2\left(1-\left(\phi_{C_2}\right)^2\right)\right)^{\frac{1}{2}}\right)^{\lambda}e^{i2\pi l^{\frac{1}{2}}\left(\left(\alpha_2\left(1-\left(\phi_{C_2}^{+im}\right)^2\right)\right)^{\frac{1}{2}}\right)^{\lambda}}, \\ l^{\frac{1}{2}}\left(\left(\alpha_2\right)^{\frac{1}{2}}r_{C_2}\right)^{\lambda}e^{i2\pi l^{\frac{1}{2}}\left(\left(\alpha_2\right)^{\frac{1}{2}}r_{C_2}^{+im}\right)^{\lambda}} \end{pmatrix}.$$

Now

$$\begin{aligned} &\left(\alpha_1^1 P_1\right)^{\lambda}\oplus\left(\alpha_2^1 P_2\right)^{\lambda} \\ =&\begin{pmatrix} \left(\frac{1}{2}\left(\begin{array}{c}\left(2^{\frac{1}{2}}\left(\left(\alpha_1\right)^{\frac{1}{2}}\eta_{C_1}\right)^{\lambda}\right)^{2} \\ +\left(2^{\frac{1}{2}}\left(\left(\alpha_2\right)^{\frac{1}{2}}\eta_{C_2}\right)^{\lambda}\right)^{2}\end{array}\right)\right)^{\frac{1}{2}}.e^{i2\pi\left(\frac{1}{2}\left(\begin{array}{c}\left(2^{\frac{1}{2}}\left(\left(\alpha_1\right)^{\frac{1}{2}}\eta_{C_1}^{+im}\right)^{\lambda}\right)^{2} \\ +\left(2^{\frac{1}{2}}\left(\left(\alpha_2\right)^{\frac{1}{2}}\eta_{C_2}^{+im}\right)^{\lambda}\right)^{2}\end{array}\right)\right)^{\frac{1}{2}}}, \\ \left(1-\frac{1}{2}\left(\begin{array}{c}\left(2^{\frac{1}{2}}\left(\left(\alpha_1\left(1-\left(\phi_{C_1}\right)^2\right)\right)^{\frac{1}{2}}\right)^{\lambda}\right)^{2} \\ +\left(2^{\frac{1}{2}}\left(\left(\alpha_2\left(1-\left(\phi_{C_2}\right)^2\right)\right)^{\frac{1}{2}}\right)^{\lambda}\right)^{2}\end{array}\right)\right)^{\frac{1}{2}}. \\ e^{i2\pi\left(1-\frac{1}{2}\left(\begin{array}{c}\left(2^{\frac{1}{2}}\left(\left(\alpha_1\left(1-\left(\phi_{C_1}^{+im}\right)^2\right)\right)^{\frac{1}{2}}\right)^{\lambda}\right)^{2} \\ +\left(2^{\frac{1}{2}}\left(\left(\alpha_2\left(1-\left(\phi_{C_2}^{+im}\right)^2\right)\right)^{\frac{1}{2}}\right)^{\lambda}\right)^{2}\end{array}\right)\right)^{\frac{1}{2}}}, \\ \left(\frac{1}{2}\left(\begin{array}{c}\left(2^{\frac{1}{2}}\left(\left(\alpha_1\right)^{\frac{1}{2}}\eta_{C_1}\right)^{\lambda}\right)^{2} \\ +\left(2^{\frac{1}{2}}\left(\left(\alpha_2\right)^{\frac{1}{2}}\eta_{C_2}\right)^{\lambda}\right)^{2}\end{array}\right)\right)^{\frac{1}{2}}.e^{i2\pi\left[\left(\frac{1}{2}\left(\begin{array}{c}\left(2^{\frac{1}{2}}\left(\left(\alpha_1\right)^{\frac{1}{2}}\eta_{C_1}^{+im}\right)^{\lambda}\right)^{2} \\ +\left(2^{\frac{1}{2}}\left(\left(\alpha_2\right)^{\frac{1}{2}}\eta_{C_2}^{+im}\right)^{\lambda}\right)^{2}\end{array}\right)\right)^{\frac{1}{2}}\right]} \end{pmatrix} \end{aligned}$$

$$
= \left(\begin{array}{c}
\left(\frac{1}{2} \left(\begin{array}{c} 2\left(\left((\alpha_1)^{\frac{1}{2}} \eta_{C_1} \right)^{\lambda} \right)^2 \\ +2\left(\left((\alpha_2)^{\frac{1}{2}} \eta_{C_2} \right)^{\lambda} \right)^2 \end{array} \right) \right)^{\frac{1}{2}} .e^{i2\pi \left(\frac{1}{2} \left(\begin{array}{c} 2\left(\left((\alpha_1)^{\frac{1}{2}} \eta_{C_1}^{+im} \right)^{\lambda} \right)^2 \\ +2\left(\left((\alpha_2)^{\frac{1}{2}} \eta_{C_2}^{+im} \right)^{\lambda} \right)^2 \end{array} \right) \right)^{\frac{1}{2}}}, \\
\left(1 - \frac{1}{2} \left(\begin{array}{c} 2\left(\left(\left(\alpha_1 \left(1 - \left(\phi_{C_1} \right)^2 \right) \right)^{\frac{1}{2}} \right)^{\lambda} \right)^2 \\ +2\left(\left(\left(\alpha_2 \left(1 - \left(\phi_{C_2} \right)^2 \right) \right)^{\frac{1}{2}} \right)^{\lambda} \right)^2 \end{array} \right) \right)^{\frac{1}{2}} . \\
e^{i2\pi \left(1 - \frac{1}{2} \left(\begin{array}{c} 2\left(\left(\left(\alpha_1 \left(1 - \left(\phi_{C_1}^{+im} \right)^2 \right) \right)^{\frac{1}{2}} \right)^{\lambda} \right)^2 \\ +2\left(\left(\left(\alpha_2 \left(1 - \left(\phi_{C_2}^{+im} \right)^2 \right) \right)^{\frac{1}{2}} \right)^{\lambda} \right)^2 \end{array} \right) \right)^{\frac{1}{2}}}, \\
\left(\frac{1}{2} \left(\begin{array}{c} 2\left(\left((\alpha_1)^{\frac{1}{2}} r_{C_1} \right)^{\lambda} \right)^2 \\ +2\left(\left((\alpha_2)^{\frac{1}{2}} r_{C_2} \right)^{\lambda} \right)^2 \end{array} \right) \right)^{\frac{1}{2}} .e^{i2\pi \left(\frac{1}{2} \left(\begin{array}{c} 2\left(\left((\alpha_1)^{\frac{1}{2}} r_{C_1}^{+im} \right)^{\lambda} \right)^2 \\ +2\left(\left((\alpha_2)^{\frac{1}{2}} r_{C_2}^{+im} \right)^{\lambda} \right)^2 \end{array} \right) \right)^{\frac{1}{2}}}
\end{array} \right)
$$

$$
= \left(\begin{array}{c}
\left(\begin{array}{c} \left(\left((\alpha_1)^{\frac{1}{2}} \eta_{C_1}^{+} \right)^{\lambda} \right)^2 + \\ \left(\left((\alpha_2)^{\frac{1}{2}} \eta_{C_2}^{+} \right)^{\lambda} \right)^2 \end{array} \right)^{\frac{1}{2}} .e^{i2\pi \left[\left(\begin{array}{c} \left(\left((\alpha_1)^{\frac{1}{2}} \eta_{C_1}^{-im} \right)^{\lambda} \right)^2 + \\ \left(\left((\alpha_2)^{\frac{1}{2}} \eta_{C_2}^{+im} \right)^{\lambda} \right)^2 \end{array} \right)^{\frac{1}{2}} \right]}, \\
\left(1 - \left(\begin{array}{c} \left(\alpha_1 \left(1 - \left(\phi_{C_1}(u) \right)^2 \right) \right)^{\lambda} \\ + \left(\alpha_2 \left(1 - \left(\phi_{C_2} \right)^2 \right) \right)^{\lambda} \end{array} \right) \right)^{\frac{1}{2}} .e^{i2\pi \left(1 - \left(\begin{array}{c} \left(\alpha_1 \left(1 - \left(\phi_{C_1}^{+im} \right)^2 \right) \right)^{\lambda} \\ + \left(\alpha_2 \left(1 - \left(\phi_{C_2}^{+im} \right)^2 \right) \right)^{\lambda} \end{array} \right) \right)^{\frac{1}{2}}} \\
\left(\begin{array}{c} \left(\left((\alpha_1)^{\frac{1}{2}} r_{C_1}^{+} \right)^{\lambda} \right)^2 + \\ \left(\left((\alpha_2)^{\frac{1}{2}} r_{C_2}^{+} \right)^{\lambda} \right)^2 \end{array} \right)^{\frac{1}{2}} .e^{i2\pi \left[\left(\begin{array}{c} \left(\left((\alpha_1)^{\frac{1}{2}} r_{C_1}^{+im} \right)^{\lambda} \right)^2 + \\ \left(\left((\alpha_2)^{\frac{1}{2}} r_{C_2}^{+im} \right)^{\lambda} \right)^2 \end{array} \right)^{\frac{1}{2}} \right]}
\end{array} \right)
$$

$$= \begin{pmatrix} \left(\sum\limits_{j=1}^{2}\left(\alpha_j\left(\eta_{C_j}^{+}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}} .e^{i2\pi\left(\sum\limits_{j=1}^{2}\left(\alpha_j\left(\eta_{C_j}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}}}, \\ \left(1-\sum\limits_{j=1}^{2}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{2}} .e^{i2\pi\left(1-\sum\limits_{j=1}^{2}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{3}}}, \\ \left(\sum\limits_{j=1}^{2}\left(\alpha_j\left(r_{C_j}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}} .e^{i2\pi\left(\sum\limits_{j=1}^{2}\left(\alpha_j\left(r_{C_j}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}}} \end{pmatrix}.$$

Let suppose that the result is true for $j = k$.

$$\left(\left(\alpha_1^1 P_1\right)^{\lambda} \oplus \left(\alpha_2^1 P_2\right)^{\lambda} \oplus \left(\alpha_3^1 P_3\right)^{\lambda} \oplus \ldots \oplus \left(\alpha_k^1 P_k\right)^{\lambda}\right)$$

$$= \begin{pmatrix} \left(\sum\limits_{j=1}^{k}\left(\alpha_j\left(\eta_{C_j}^{+}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}} .e^{i2\pi\left(\sum\limits_{j=1}^{k}\left(\alpha_j\left(\eta_{C_j}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}}}, \\ \left(1-\sum\limits_{j=1}^{k}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{2}} .e^{i2\pi\left(1-\sum\limits_{j=1}^{k}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{3}}}, \\ \left(\sum\limits_{j=1}^{k}\left(\alpha_j\left(r_{C_j}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}} .e^{i2\pi\left(\sum\limits_{j=1}^{k}\left(\alpha_j\left(r_{C_j}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}}} \end{pmatrix}.$$

We shows that the result is true for $j = k + 1$. So

$$\left(\left(\alpha_1 P_1\right)^{\lambda} \oplus \left(\alpha_2 P_2\right)^{\lambda} \oplus \left(\alpha_3 P_3\right)^{\lambda} \oplus \ldots \oplus \left(\alpha_k P_k\right)^{\lambda}\right) \oplus \left(\alpha_{k+1} P_{k+1}\right)^{\lambda}$$

$$= \begin{pmatrix} \left(\sum\limits_{j=1}^{k}\left(\alpha_j\left(\eta_{C_j}^{+}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}} .e^{i2\pi\left(\sum\limits_{j=1}^{k}\left(\alpha_j\left(\eta_{C_j}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}}}, \\ \left(1-\sum\limits_{j=1}^{k}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{2}} .e^{i2\pi\left(1-\sum\limits_{j=1}^{k}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{3}}}, \\ \left(\sum\limits_{j=1}^{k}\left(\alpha_j\left(r_{C_j}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}} .e^{i2\pi\left(\sum\limits_{j=1}^{k}\left(\alpha_j\left(r_{C_j}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}}} \end{pmatrix} \oplus$$

$$\left(\begin{pmatrix} l^{\frac{1}{2}}\left(\left(\alpha_{k+1}\right)^{\frac{1}{2}}\eta_{C_{k+1}}\right)^{\lambda} e^{i2\pi l^{\frac{1}{2}}\left(\left(\alpha_{k+1}\right)^{\frac{1}{2}}\eta_{C_{k+1}}^{+im}\right)^{\lambda}}, \\ l^{\frac{1}{2}}\left(\left(\alpha_{k+1}\left(1-\left(\phi_{C_{k+1}}\right)^2\right)\right)^{\frac{1}{2}}\right)^{\lambda} e^{i2\pi l^{\frac{1}{2}}\left(\left(\alpha_{k+1}\left(1-\left(\phi_{C_{k+1}}^{+im}\right)^2\right)\right)^{\frac{1}{2}}\right)^{\lambda}}, \\ l^{\frac{1}{2}}\left(\left(\alpha_{k+1}\right)^{\frac{1}{2}} r_{C_{k+1}}\right)^{\lambda} e^{i2\pi l^{\frac{1}{2}}\left(\left(\alpha_1\right)^{\frac{1}{2}} r_{C_{k+1}}^{+im}\right)^{\lambda}} \end{pmatrix}\right)$$

$$= \left(\begin{array}{c} \left(\sum_{j=1}^{k+1}\left(\alpha_j\left(\eta_{C_j}^{+}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}} .e^{i2\pi\left(\sum_{j=1}^{k+1}\left(\alpha_j\left(\eta_{C_j}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}}}, \\ \left(1-\sum_{j=1}^{k+1}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{2}} .e^{i2\pi\left(1-\sum_{j=1}^{k+1}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{3}}}, \\ \left(\sum_{j=1}^{k+1}\left(\alpha_j\left(r_{C_j}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}} .e^{i2\pi\left(\sum_{j=1}^{k+1}\left(\alpha_j\left(r_{C_j}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{2}}} \end{array} \right).$$

Thus

$$\left(\left((\alpha_1P_1)^{\lambda}\oplus(\alpha_2P_2)^{\lambda}\oplus(\alpha_3P_3)^{\lambda}\oplus...\oplus(\alpha_kP_k)^{\lambda}\right)\oplus(\alpha_{k+1}P_{k+1})^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$= \left(\begin{array}{c} \left(\left(\sum_{j=1}^{k+1}\left(\alpha_j\left(\eta_{C_j}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left(\left(\sum_{r=1}^{k+1}\left(\alpha_j\left(\eta_{C_j}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}}, \\ \left(\left(1-\sum_{j=1}^{k+1}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left(\left(1-\sum_{j=1}^{k+1}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}}, \\ \left(\left(\sum_{j=1}^{k+1}\left(\alpha_j\left(r_{C_j}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left(\left(\sum_{r=1}^{k+1}\left(\alpha_j\left(r_{C_j}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}} \end{array} \right).$$

☐

Theorem 2. *Let* $\left\{P_j=\left(\eta_{C_j}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_j}e^{i2\pi\phi_{C_r}^{+im}},r_{C_j}e^{i2\pi r_{C_j}^{+im},}\right):j=1,2,...,m\right\}$ *be the collection of* C_rCT*-SPF values values. Then the* C_r^2C*-*$PFWAM_E$ *operator is defined by*

$$C_r^2C-PPFWAM_E\left(P_1,P_2,P_3,...,P_m\right)$$

$$= \left(\begin{array}{c} \left(\left(\sum_{r=1}^{m}\left(\alpha_j\left(\eta_{C_j}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left(\left(\sum_{j=1}^{m}\left(\alpha_r\left(\eta_{C_j}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}}, \\ \left(\left(1-\sum_{j=1}^{m}\left(\alpha_j\left(1-\left(\phi_{C_j}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).{\scriptstyle\frac{1}{2}}e^{i2\pi\left(\left(1-\sum_{j=1}^{m}\left(\alpha_j\left(1-\left(\phi_{C_j}^{+im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}}, \\ \left(\left(1-\sum_{j=1}^{m}\left(\alpha_j\left(1-\left(r_{C_j}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_j\left(1-\left(r_{C_j}^{+im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}} \end{array} \right).$$

$E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_n\right)^T$ *are a weight vectors of* P_r *with* $\alpha_r\in[0,1]$ *and* $\Sigma_{r=1}^{n}\alpha_r=1,r=1,2,...,m.$

Proof. Similar to the proof of Theorem 1. ☐

Theorem 3. *Let* $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ *be the collection of* C_rC*-*PF *values. Let* $E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ *are a weight vectors of* P_r *with* $\alpha_r\in[0,1]$ *and* $\Sigma_{r=1}^{m}\alpha_r=1$*. If* $\left(\eta_{C_1}e^{i2\pi\eta_{C_1}^{+im}},\phi_{C_1}e^{i2\pi\phi_{C_1}^{+im}},r_{C_1}e^{i2\pi r_{C_1}^{+im},}\right)=$

Proposition 1. $\left(\eta_{C_2}e^{i2\pi\eta_{C_2}^{+im}},\phi c_2 e^{i2\pi\phi_{C_2}^{+im}},r_{C_2}e^{i2\pi r_{C_2}^{+im}},\right)$
$=\left(\eta_{C_m}e^{i2\pi\eta_{C_m}^{+im}},\phi_{C_m}e^{i2\pi\phi_{C_m}^{+im}},r_{C_m}e^{i2\pi r_{C_m}^{+im}},\right)$
$=\left(\eta_C e^{i2\pi\eta_C^{+im}},\phi_C e^{i2\pi\phi_C^{+im}},r_C e^{i2\pi r_C^{+im}},\right)$ *and* $\lambda=1$,
then $C_r^1C\text{-}PFWAM_E\left(P_1,P_2,P_3,...,P_m\right)$
$=\left(\eta_C e^{i2\pi\eta_C^{+im}},\phi_C e^{i2\pi\phi_C^{+im}},r_C e^{i2\pi r_C^{+im}},\right)$.

Proof. Let $\left(\eta_{C_1}e^{i2\pi\eta_{C_1}^{+im}},\phi_{C_1}e^{i2\pi\phi_{C_1}^{+im}},r_{C_1}e^{i2\pi r_{C_1}^{+im}},\right)$,
$=\left(\eta_{C_2}e^{i2\pi\eta_{C_2}^{+im}},\phi c_2 e^{i2\pi\phi_{C_2}^{+im}},r_{C_2}e^{i2\pi r_{C_2}^{+im}},\right)$
$=\left(\eta_{C_m}e^{i2\pi\eta_{C_m}^{+im}},\phi_{C_m}e^{i2\pi\phi_{C_m}^{+im}},r_{C_m}e^{i2\pi r_{C_m}^{+im}},\right)$
$=\left(\eta_C e^{i2\pi\eta_C^{+im}},\phi_C e^{i2\pi\phi_C^{+im}},r_C e^{i2\pi r_C^{+im}},\right)$ and $E=(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^m\alpha_r=1$, where $r=1,2,...,m$. Based on Definition 7, we get

$$C_rC-PFWAM_E\left(P_1,P_2,P_3,...,P_m\right)$$

$$=\left(\begin{array}{c}\left(\left(\sum_{r=1}^m\left(\alpha_r\left(\eta_{C_r}\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}e^{i2\pi\left(\left(\sum_{r=1}^m\left(\alpha_r\left(\eta_{C_r}^{+im}\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}},\\ \left(\left(1-\sum_{r=1}^m\left(\alpha_r\left(1-\left(\phi_{C_r}\right)^2\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}e^{i2\pi\left(\left(1-\sum_{r=1}^m\left(\alpha_r\left(1-\left(\phi_{C_r}^{+im}\right)^2\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}},\\ \left(\left(\sum_{r=1}^m\left(\alpha_r\left(r_{C_r}\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}e^{i2\pi\left(\left(\sum_{r=1}^m\left(\alpha_r\left(r_{C_r}^{+im}\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}}\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\sum_{r=1}^m\left(\alpha_r\left(\eta_C\right)^2\right)\right)^{\frac{1}{2}}.e^{i2\pi\left(\sum_{r=1}^m\left(\alpha_r\left(\eta_C^{+im}\right)^2\right)\right)^{\frac{1}{2}}},\\ \left(1-\sum_{r=1}^m\left(\alpha_r\left(1-\left(\phi_C\right)^2\right)\right)\right)^{\frac{1}{2}}e^{i2\pi\left(1-\sum_{r=1}^m\left(\alpha_r\left(1-\left(\phi_C^{+im}\right)^2\right)\right)\right)^{\frac{1}{2}}},\\ \left(\sum_{r=1}^m\left(\alpha_r\left(r_C\right)^2\right)\right)^{\frac{1}{2}}.e^{i2\pi\left(\sum_{r=1}^m\left(\alpha_r\left(r_C^{+im}\right)^2\right)\right)^{\frac{1}{2}}}\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\left(\eta_C\right)^2\sum_{r=1}^m\left(\alpha_r\right)\right)^{\frac{1}{2}}.e^{i2\pi\left(\left(\eta_C^{+im}\right)^2\sum_{r=1}^m\left(\alpha_r\right)\right)^{\frac{1}{2}}},\\ \left(1-\left(1-\left(\phi_C\right)^2\right)\sum_{r=1}^m\left(\alpha_r\right)\right)^{\frac{1}{2}}.e^{i2\pi\left(1-\left(1-\left(\phi_C^{+im}\right)^2\right)\sum_{r=1}^m\left(\alpha_r\right)\right)^{\frac{1}{2}}},\\ \left(\left(r_C\right)^2\sum_{r=1}^m\left(\alpha_r\right)\right)^{\frac{1}{2}}.e^{i2\pi\left(\left(r_C^{+im}\right)^2\sum_{r=1}^m\left(\alpha_r\right)\right)^{\frac{1}{2}}}\end{array}\right)$$

$$\begin{aligned} &= \left(\begin{array}{c} \left((\eta_C)^2\right)^{\frac{1}{2}} e^{i2\pi\left(\left(\eta_C^{+im}\right)^2\right)^{\frac{1}{2}}}, \\ \left(1-\left(1-(\phi_C)^2\right)\right)^{\frac{1}{2}} .e^{i2\pi\left(1-\left(1-\left(\phi_C^{+im}\right)^2\right)\right)^{\frac{1}{2}}}, \\ \left((r_C)^2\right)^{\frac{1}{2}} e^{i2\pi\left(\left(r_C^{+im}\right)^2\right)^{\frac{1}{2}}} \end{array}\right) \\ &= \left(\eta_C.e^{i2\pi\eta_C^{+im}}, \phi_C.e^{i2\pi\phi_C^{+im}}, r_C.e^{i2\pi r_C^{+im}}\right). \end{aligned}$$

□

Theorem 4. *Let* $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* $C_rC\text{-}PF$ *values. Let* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0, 1]$ *and* $\Sigma_{r=1}^m \alpha_r = 1$. *If* $\left(\eta_{C_1}e^{i2\pi\eta_{C_1}^{+im}}, \phi_{C_1}e^{i2\pi\phi_{C_1}^{+im}}, r_{C_1}e^{i2\pi r_{C_1}^{+im},}\right)$

Proposition 2. $= \left(\eta_{C_2}e^{i2\pi\eta_{C_2}^{+im}}, \phi c_2 e^{i2\pi\phi_{C_2}^{+im}}, r_{C_2}e^{i2\pi r_{C_2}^{+im},}\right)$
$= \left(\eta_{C_m}e^{i2\pi\eta_{C_m}^{+im}}, \phi_{C_m}e^{i2\pi\phi_{C_m}^{+im}}, r_{C_m}e^{i2\pi r_{C_m}^{+im},}\right)$
$= \left(\eta_C e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, r_C e^{i2\pi r_C^{+im},}\right)$ *and* $\lambda = 1$, *then*
$C_r^2C\text{-}PFWAM_E\left(P_1, P_2, P_3, ..., P_m\right)$
$= \left(\eta_C e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, r_C e^{i2\pi r_C^{+im},}\right).$

Proof. Similar to the proof of Theorem 3 □

Theorem 5. *Let* $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* $C_rC\text{-}PF$ *values and* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0, 1]$ *and* $\Sigma_{r=1}^m \alpha_r = 1$, *where* $r = 1, 2, ..., m$. *Let*

$$P^- = \left(\begin{array}{c} \min\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi \min\limits_{1\le r\le m} \eta_{C_r}^{+im}}, \max\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \phi_{C_r}^{+im}}, \\ , \min\limits_{1\le r\le m} r_{C_r}.e^{i2\pi \min\limits_{1\le r\le m} r_{C_r}^{+im}} \end{array}\right)$$

$$P^+ = \left(\begin{array}{c} \max\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \eta_{C_r}^{+im}}, \min\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \phi_{C_r}^{+im}}, \\ , \max\limits_{1\le r\le m} r_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} r_{C_r}^{+im}}, \end{array}\right),$$ *where and* $\lambda = 1$. *Then* $P^- \le C_r^1C\text{-}FFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \le P^+$.

Proof. Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of $C_rC\text{-}PF$ values and $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r, where $r = 1, 2, ..., m$ with $\alpha_r \in [0, 1]$ and $\Sigma_{r=1}^m \alpha_r = 1$. Let

$$P^- = \left(\begin{array}{c} \min\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi \min\limits_{1\le r\le m} \eta_{C_r}^{+im}}, \max\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \phi_{C_r}^{+im}}, \\ , \min\limits_{1\le r\le m} r_{C_r}.e^{i2\pi \min\limits_{1\le r\le m} r_{C_r}^{+im}} \end{array}\right)$$

$$P^+ = \left(\begin{array}{c} \max\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \eta_{C_r}^{+im}}, \min\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} \phi_{C_r}^{+im}}, \\ , \max\limits_{1\le r\le m} r_{C_r}.e^{i2\pi \max\limits_{1\le r\le m} r_{C_r}^{+im}}, \end{array}\right)$$

where and $\lambda = 1$, we get $\left(\eta_{C_t}\right)^2 \leq \left(\max\limits_{1\leq r\leq m} \eta_{C_r}\right)^2$ where $t = 1, 2, ..., m$. Then we have $\alpha_r \left(\eta_{C_t}\right)^2 \leq \alpha_r \left(\max\limits_{1\leq r\leq m} \eta_{C_r}\right)^2$. This implies that $\sum\limits_{r=1}^{m} \alpha_r \left(\eta_{C_t}\right)^2 \leq \sum\limits_{r=1}^{m} \alpha_r \left(\max\limits_{1\leq r\leq m} \eta_{C_r}\right)^2$ then we have, $\sum\limits_{r=1}^{m} \alpha_r \left(\eta_{C_t}\right)^2 \leq \left(\max\limits_{1\leq r\leq m} \eta_{C_r}\right)^2 \sum\limits_{r=1}^{m} \alpha_r$ This implies that $\sum\limits_{r=1}^{m} \alpha_r \left(\eta_{C_t}\right)^2 \leq \left(\max\limits_{1\leq r\leq m} \eta_{C_r}\right)^2$ and this implies that $\left(\sum\limits_{r=1}^{m} \alpha_r \left(\eta_{C_t}\right)^2\right)^{\frac{1}{2}} \leq \max\limits_{1\leq r\leq m} \eta_{C_r}$. Hence

$\left(\sum\limits_{r=1}^{m} \alpha_r \left(\eta_{C_t}\right)^2\right)^{\frac{1}{2}} \leq \max\limits_{1\leq r\leq m} \eta_{C_r}$. Also

$\left(\sum\limits_{r=1}^{m} \alpha_r \left(\eta_{C_t}^{im}\right)^2\right)^{\frac{1}{2}} \leq \max\limits_{1\leq r\leq m} \eta_{C_r}^{im}$. This implies that

$e^{i2\pi\left(\sum\limits_{r=1}^{m} \alpha_r \left(\eta_{C_t}^{im}\right)^2\right)^{\frac{1}{2}}} \leq e^{i2\pi \max\limits_{1\leq r\leq m} \eta_{C_r}^{im}}$. Thus

$$\begin{aligned} & \left(\sum_{r=1}^{m} \alpha_r \left(\eta_{C_t}^{im}\right)^2\right)^{\frac{1}{2}} . e^{i2\pi\left(\sum\limits_{r=1}^{m} \alpha_r \left(\eta_{C_t}^{im}\right)^2\right)^{\frac{1}{2}}} \\ \leq \; & \max_{1\leq r\leq m} \eta_{C_r}^{im} . e^{i2\pi \max\limits_{1\leq r\leq m} \eta_{C_r}^{im}}. \end{aligned}$$

On the other hand, we get $\left(\phi_{C_r}\right)^2 \geq \left(\min\limits_{1\leq r\leq m} \phi_C\right)^2$. This implies that $-\left(\phi_{C_r}\right)^2 \leq -\left(\min\limits_{1\leq r\leq m} \phi_C\right)^2$ and , also $1 - \left(\phi_{C_r}\right)^2 \leq 1 - \left(\min\limits_{1\leq r\leq m} \phi_C\right)^2$, where, $r = 1, 2, ..., m$. Then we have $\alpha_r \left(1 - \left(\phi_{C_r}(u)\right)^2\right) \leq \alpha_r \left(1 - \left(\phi_C(u)\right)^2\right)$ $\alpha_r \left(1 - \left(\phi_{C_r}\right)^2\right) \leq \alpha_r \left(1 - \left(\min\limits_{1\leq r\leq m} \phi_C\right)^2\right)$ then we get,

$$\begin{aligned} \alpha_r \left(1 - \left(\phi_{C_r}\right)^2\right) &\leq \alpha_r \left(1 - \left(\min_{1\leq r\leq m} \phi_C\right)^2\right) \\ &\Rightarrow \sum_{r=1}^{m} \alpha_r \left(1 - \left(\phi_{C_r}\right)^2\right) \leq \sum_{r=1}^{m} \alpha_r \left(1 - \left(\min_{1\leq r\leq m} \phi_C\right)^2\right) \\ &\Rightarrow \sum_{r=1}^{m} \alpha_r \left(1 - \left(\phi_{C_r}\right)^2\right) \leq \left(1 - \left(\min_{1\leq r\leq m} \phi_C\right)^2\right) \sum_{r=1}^{m} \alpha_r \\ &\Rightarrow -\sum_{r=1}^{m} \alpha_r \left(1 - \left(\phi_{C_r}\right)^2\right) \geq -\left(1 - \left(\min_{1\leq r\leq m} \phi_C\right)^2\right) \\ &\Rightarrow 1 - \sum_{r=1}^{m} \alpha_r \left(1 - \left(\phi_{C_r}\right)^2\right) \geq 1 - \left(1 - \left(\min_{1\leq r\leq m} \phi_C\right)^2\right) \\ &\Rightarrow 1 - \sum_{r=1}^{m} \alpha_r \left(1 - \left(\phi_{C_r}\right)^2\right) \geq \left(\min_{1\leq r\leq m} \phi_C\right)^2 \\ &\Rightarrow \left(1 - \sum_{r=1}^{m} \alpha_r \left(1 - \left(\phi_{C_r}\right)^2\right)\right)^{\frac{1}{2}} \geq \min_{1\leq r\leq m} \phi_C. \end{aligned}$$

Also $\left(1-\sum\limits_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}\right)^2\right)\right)^{\frac{1}{2}} \geq \min\limits_{1\leq r\leq m}\phi_C$. This implies that

$\left(1-\sum\limits_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}\right)^2\right)\right)^{\frac{1}{2}} \geq \min\limits_{1\leq r\leq m}\phi_C$. Also

$\left(1-\sum\limits_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)^2\right)\right)^{\frac{1}{2}} \geq \min\limits_{1\leq r\leq m}\phi_C^{+im}$. This implies that

$e^{i2\pi\left(1-\sum\limits_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}^{+im}\right)^2\right)\right)^{\frac{1}{2}}} \geq e^{i2\pi\min\limits_{1\leq r\leq m}\phi_C^{+im}}$. Thus

$$\begin{aligned}&\left(1-\sum_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}\right)^2\right)\right)^{\frac{1}{2}}.e^{i2\pi\left(1-\sum\limits_{r=1}^{m}\alpha_r\left(1-\left(\phi_{C_r}^{+im}\right)^2\right)\right)^{\frac{1}{2}}}\\ \geq\ &\min_{1\leq r\leq m}\phi_C.e^{i2\pi\min\limits_{1\leq r\leq m}\phi_C^{-im}}.\end{aligned}$$

On the other hand, $\left(r_{C_t}\right)^2 \leq \left(\max\limits_{1\leq r\leq m}r_{C_r}\right)^2$ where $t = 1, 2, ..., m$. Then we have $r_r\left(\eta_{C_t}\right)^2 \leq r_r\left(\max\limits_{1\leq r\leq m}\eta_{C_r}\right)^2$. This implies that $\sum\limits_{r=1}^{m}\alpha_r\left(r_{C_t}\right)^2 \leq \sum\limits_{r=1}^{m}\alpha_r\left(\max\limits_{1\leq r\leq m}r_{C_r}\right)^2$ then we have, $\sum\limits_{r=1}^{m}\alpha_r\left(r_{C_t}\right)^2 \leq \left(\max\limits_{1\leq r\leq m}r_{C_r}\right)^2\sum\limits_{r=1}^{m}\alpha_r$ This implies that $\sum\limits_{r=1}^{m}\alpha_r\left(r_{C_t}\right)^2 \leq \left(\max\limits_{1\leq r\leq m}r_{C_r}\right)^2$ and this implies that $\left(\sum\limits_{r=1}^{m}\alpha_r\left(r_{C_t}\right)^2\right)^{\frac{1}{2}} \leq \max\limits_{1\leq r\leq m}r_{C_r}$. Hence

$\left(\sum\limits_{r=1}^{m}\alpha_r\left(r_{C_t}\right)^2\right)^{\frac{1}{2}} \leq \max\limits_{1\leq r\leq m}r_{C_r}$. Also

$\left(\sum\limits_{r=1}^{m}\alpha_r\left(r_{C_t}^{im}\right)^2\right)^{\frac{1}{2}} \leq \max\limits_{1\leq r\leq m}r_{C_r}^{im}$. This implies that

$e^{i2\pi\left(\sum\limits_{r=1}^{m}\alpha_r\left(r_{C_t}^{im}\right)^2\right)^{\frac{1}{2}}} \leq e^{i2\pi\max\limits_{1\leq r\leq m}r_{C_r}^{im}}$. Thus

$$\begin{aligned}&\left(\sum_{r=1}^{m}\alpha_r\left(r_{C_t}^{im}\right)^2\right)^{\frac{1}{2}}.e^{i2\pi\left(\sum\limits_{r=1}^{m}\alpha_r\left(r_{C_t}^{im}\right)^2\right)^{\frac{1}{2}}}\\ \leq\ &\max_{1\leq r\leq m}r_{C_r}^{im}.e^{i2\pi\max\limits_{1\leq r\leq m}r_{C_r}^{im}}.\end{aligned}$$

Based on Definition **??** and Definition 7, we get CrC-$PFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \leq P^+$. Similarly, we can have C_r^1C-$PFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \geq P^-$. Finally we get,

$$P^+ \geq C_r^1C - PFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \geq P^-.$$

□

Theorem 6. *Let* $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* C_r^2C-PF *values and* $E = \left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0, 1]$ *and* $\Sigma_{r=1}^{m}\alpha_r = 1$, *where* $r = 1, 2, ..., m$. *Let*

$P^- = \left(\min\limits_{1\leq r\leq m}\eta_{C_r}.e^{i2\pi\min\limits_{1\leq r\leq m}\eta_{C_r}^{+im}}, \max\limits_{1\leq r\leq m}\phi_{C_r}.e^{i2\pi\max\limits_{1\leq r\leq m}\phi_{C_r}^{+im}}, \max\limits_{1\leq r\leq m}r_{C_r}.e^{i2\pi\min\limits_{1\leq r\leq m}r_{C_r}^{+im}}\right)$

$P^{+} = \left(\begin{array}{c} \max\limits_{1\leq r\leq m} \eta_{C_r}.e^{i2\pi \max\limits_{1\leq r\leq m} \eta_{C_r}^{+im}}, \min\limits_{1\leq r\leq m} \phi_{C_r}.e^{i2\pi \max\limits_{1\leq r\leq m} \phi_{C_r}^{+im}}, \\ , \min\limits_{1\leq r\leq m} r_{C_r}.e^{i2\pi \max\limits_{1\leq r\leq m} r_{C_r}^{+im}}, \end{array}\right)$, *where and* $\lambda = 1$. *Then* $P^{-} \leq C_r^2 C\text{-}FFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \leq P^{+}$.

Proof. Similar to the proof of Theorem 5. ☐

Theorem 7. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ *and* $\left\{P_r^{*} = \left(\eta_{C_r}^{*} e^{i2\pi\eta_{C_r}^{+im*}}, \phi_{C_r}^{*} e^{i2\pi\phi_{C_r}^{+im*}}, r_{C_r}^{*} e^{i2\pi r_{C_r}^{+im*}}\right) : r = 1, 2, ..., m\right\}$ *are two collections of* $C_rC\text{-}PF$ *values. If* $\eta_{C_r} \leq \eta_{C_r}^{*}$, $\eta_{C_r}^{im} \leq \eta_{C_r}^{im*}$, $\phi_{C_r} \geq \phi_{C_r}^{*}$, $\phi_{C_r}^{im} \geq \phi_{C_r}^{im*}, r_{C_r} \leq r_{C_r}^{*}$, *and* $r_{C_r}^{im} \leq r_{C_r}^{im*}$ *where* $r = 1, 2, ..., m$, *then* $C_r^1 C\text{-}PFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \leq C_r^1 C\text{-}PFWAM_E\left(P_1^{*}, P_2^{*}, P_3^{*}, ..., P_m^{*}\right)$.

Proof. Given that $P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right)$ and $P_r^{*} = \left(\eta_{C_r}^{*} e^{i2\pi\eta_{C_r}^{+im*}}, \phi_{C_r}^{*} e^{i2\pi\phi_{C_r}^{+im*}}, r_{C_r}^{*} e^{i2\pi r_{C_r}^{+im*}}\right)$ are two collections of $C_rC\text{-}PF$ values, where $r = 1, 2, ..., m$. If $\eta_{C_r} \leq \eta_{C_r}^{*}$, $\eta_{C_r}^{im} \leq \eta_{C_r}^{im*}$, $\phi_{C_r} \geq \phi_{C_r}^{*}$, $\phi_{C_r}^{im} \geq \phi_{C_r}^{im*}, r_{C_r} \leq r_{C_r}^{*}$, and $r_{C_r}^{im} \leq r_{C_r}^{im*}$ then $\left(\eta_{C_r}\right)^2 \leq \left(\eta_{C_r}^{*}\right)^2$. As we have

$$\begin{aligned} \left(\eta_{C_r}\right)^2 &\leq \left(\eta_{C_r}^{*}\right)^2 \\ &\Rightarrow \alpha_r \left(\eta_{C_r}\right)^2 \leq \alpha_r \left(\eta_{C_r}^{*}\right)^2 \\ &\Rightarrow \left(\alpha_r \left(\eta_{C_r}\right)^2\right)^{\lambda} \leq \left(\alpha_r \left(\eta_{C_r}^{*}\right)^2\right)^{\lambda} \\ &\Rightarrow \sum_{r=1}^{m} \left(\alpha_r \left(\eta_{C_r}\right)^2\right)^{\lambda} \leq \left(\alpha_r \left(\eta_{C_r}^{*}\right)^2\right)^{\lambda} \end{aligned}$$

$$\begin{aligned} &\Rightarrow \left(\sum_{r=1}^{m} \left(\alpha_r \left(\eta_{C_r}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^{m} \left(\alpha_r \left(\eta_{C_r}^{*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}} \\ &\Rightarrow \left(\left(\sum_{r=1}^{m} \left(\alpha_r \left(\eta_{C_r}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} \leq \left(\left(\sum_{r=1}^{m} \left(\alpha_r \left(\eta_{C_r}^{*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}. \end{aligned}$$

Thus

$$\left(\left(\sum_{r=1}^{m} \left(\alpha_r \left(\eta_{C_r}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} \leq \left(\left(\sum_{r=1}^{m} \left(\alpha_r \left(\eta_{C_r}^{*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}. \quad \text{(i)}$$

Further,

$$\begin{aligned} \left(\eta_{C_r}^{im}(u)\right)^2 &\leq \left(\eta_{C_r}^{im*}(u)\right)^2 \leq \\ &\Rightarrow \alpha_r \left(\eta_{C_r}^{im}(u)\right)^2 \leq \alpha_r \left(\eta_{C_r}^{im*}(u)\right)^2 \\ &\Rightarrow \left(\alpha_r \left(\eta_{C_r}^{im}(u)\right)^2\right)^{\lambda} \leq \left(\alpha_r \left(\eta_{C_r}^{im*}(u)\right)^2\right)^{\lambda} \\ &\Rightarrow \sum_{r=1}^{m} \left(\alpha_r \left(\eta_{C_r}^{im}(u)\right)^2\right)^{\lambda} \leq \left(\alpha_r \left(\eta_{C_r}^{im*}(u)\right)^2\right)^{\lambda} \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}} \\
&\Rightarrow \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{im*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}} \leq e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}}}. \qquad \text{(ii)}$$

From (i) and (ii)

$$\begin{aligned}
&\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}} \\
&\leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}}\right]}. \qquad (A)
\end{aligned}$$

Now

$$\begin{aligned}
\left(\phi_{C_r}\right)^2 &\geq \left(\phi_{C_r}^{*}\right)^2 \\
&\Rightarrow -\left(\phi_{C_r}\right)^2 \leq -\left(\phi_{C_r}^{*}\right)^2 \\
&\Rightarrow \left(1-\left(\phi_{C_r}\right)^2\right) \leq \left(1-\left(\phi_{C_r}^{*}\right)^2\right) \\
&\Rightarrow \alpha_r\left(1-\left(\phi_{C_r}\right)^2\right) \leq \alpha_r\left(1-\left(\phi_{C_r}^{*}\right)^2\right) \\
&\Rightarrow \left(\alpha_r\left(1-\left(\phi_{C_r}\right)^2\right)\right)^{\lambda} \leq \left(\alpha_r\left(1-\left(\phi_{C_r}^{*}\right)^2\right)\right)^{\lambda}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}\right)^2\right)\right)^{\lambda} \\
\leq\ &\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{*}\right)^2\right)\right)^{\lambda} \\
\Rightarrow\ &-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}\right)^2\right)\right)^{\lambda} \geq -\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{*}\right)^2\right)\right)^{\lambda} \\
\Rightarrow\ &1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}\right)^2\right)\right)^{\lambda} \geq 1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{*}\right)^2\right)\right)^{\lambda} \\
\Rightarrow\ &\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \geq \left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{*}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \\
\Rightarrow\ &\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} \geq \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{*}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}. \qquad \text{(iii)}
\end{aligned}$$

Also

$$\begin{aligned}
& \sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)^2\right)\right)^{\lambda} \\
\leq\ & \sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im*}\right)^2\right)\right)^{\lambda} \\
\Rightarrow\ & -\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)^2\right)\right)^{\lambda} \geq -\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im*}\right)^2\right)\right)^{\lambda} \\
\Rightarrow\ & 1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)^2\right)\right)^{\lambda} \geq 1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im*}\right)^2\right)\right)^{\lambda}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ & \left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \geq \left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im*}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}} \\
\Rightarrow\ & \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} \geq \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im*}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}.
\end{aligned}$$

This implies that

$$e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]} \geq e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im*}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]}. \qquad \mathrm{(iv)}$$

From $\mathrm{(iii)}$ and $\mathrm{(iv)}$, we get

$$\begin{aligned}
& \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1\left(\phi_{C_r}^{+im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]} \\
\geq & \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{+im*}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]}. \qquad (B).
\end{aligned}$$

Now

$$\begin{aligned}
\left(r_{C_r}\right)^2 &\leq \left(r_{C_r}^{*}\right)^2 \\
&\Rightarrow \alpha_r\left(r_{C_r}\right)^2 \leq \alpha_r\left(r_{C_r}^{*}\right)^2 \\
&\Rightarrow \left(\alpha_r\left(r_{C_r}\right)^2\right)^{\lambda} \leq \left(\alpha_r\left(r_{C_r}^{*}\right)^2\right)^{\lambda} \\
&\Rightarrow \sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)^2\right)^{\lambda} \leq \left(\alpha_r\left(r_{C_r}^{*}\right)^2\right)^{\lambda}
\end{aligned}$$

$$
\begin{aligned}
&\Rightarrow \left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}} \\
&\Rightarrow \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Thus

$$
\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}. \qquad \text{(vii)}
$$

Further,

$$
\begin{aligned}
\left(r_{C_r}^{im}(u)\right)^2 \quad &\leq \quad \left(r_{C_r}^{im*}(u)\right)^2 \leq \\
&\Rightarrow \alpha_r\left(r_{C_r}^{im}(u)\right)^2 \leq \alpha_r\left(r_{C_r}^{im*}(u)\right)^2 \\
&\Rightarrow \left(\alpha_r\left(r_{C_r}^{im}(u)\right)^2\right)^{\lambda} \leq \left(\alpha_r\left(r_{C_r}^{im*}(u)\right)^2\right)^{\lambda} \\
&\Rightarrow \sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{im}(u)\right)^2\right)^{\lambda} \leq \left(\alpha_r\left(r_{C_r}^{im*}(u)\right)^2\right)^{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{im}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{im*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}} \\
&\Rightarrow \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{im}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} \leq \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{im*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Hence

$$
e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}} \leq e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}}. \qquad \text{(viii)}
$$

From (i) and (ii)

$$
\begin{aligned}
&\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} . e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}} \\
\leq &\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} . e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]}. \qquad (D)
\end{aligned}
$$

Thus, from (A) ,(B) ,(C) and (D) , we have

$$\left(\begin{array}{c} \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}}, \\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1\left(\phi_{C_r}^{+im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]}, \\ \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}} \end{array}\right)$$

$$\leq\left(\begin{array}{c} \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_r}^{+im*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}}\right]}, \\ \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_r}^{+im*}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]}, \\ \left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}} .e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im*}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]} \end{array}\right).$$

Hence $CrC\text{-}PFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \leq CrC\text{-}FFWAM_E\left(P_1^*, P_2^*, P_3^*, ..., P_m^*\right).$ ☐

Theorem 8. *Let* $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r=1,2,...,m\right\}$ *and* $\left\{P_r^*=\left(\eta_{C_r}^*e^{i2\pi\eta_{C_r}^{+im*}}, \phi_{C_r}^*e^{i2\pi\phi_{C_r}^{+im*}}, r_{C_r}^*e^{i2\pi r_{C_r}^{+im*}}\right) : r=1,2,...,m\right\}$ *are two collections of* $C_rC\text{-}PF$ *values. If* $\eta_{C_r} \leq \eta_{C_r}^*$, $\eta_{C_r}^{im} \leq \eta_{C_r}^{im*}$, $\phi_{C_r} \geq \phi_{C_r}^*$, $\phi_{C_r}^{im} \geq \phi_{C_r}^{im*}$, $r_{C_r} \geq r_{C_r}^*$, *and* $r_{C_r}^{im} \geq r_{C_r}^{im*}$ *where* $r=1,2,...,m$, *then* $C_r^2C\text{-}PFWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \leq C_r^2C\text{-}PFWAM_E\left(P_1^*, P_2^*, P_3^*, ..., P_m^*\right).$

Proof. Similar to the proof of Theorem 7. ☐

Definition 8. *Let* $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r=1,2,...,m\right\}$ *be the collection of* $C_rC\text{-}PF$ *values and let* $C_r^1C\text{-}PFOWAM : \Omega^m \rightarrow \Omega$, *if*
$C_r^1C\text{-}PFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) =$
$\left(\left(\alpha_1P_{\delta(1)}\right)^{\lambda} \oplus \left(\alpha_2P_{\delta(1)}\right)^{\lambda} \oplus \left(\alpha_3P_{\delta(3)}\right)^{\lambda} \oplus ... \oplus \left(\alpha_mP_{\delta(m)}\right)^{\lambda}\right)^{\frac{1}{\lambda}}$ *then* $C_r^1C\text{-}PFOWAM$ *is called a Circular Complex fermatean fuzzy ordered weighted averaging mean operator of dimension* n, *where* $\left(\delta\left(1\right), \delta\left(2\right), ..., \delta\left(m\right)\right)$ *is a permutation of* $\left(1, 2, ..., m\right)$ *such that* $P_{\delta(r-1)} \geq P_{\delta(r)}$ *for all* r, Ω *is the set of all* $C_rC\text{-}PF$ *values,* $E = \left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0,1]$ *and* $\Sigma_{r=1}^{m}\alpha_r = 1$, *where* $r=1,2,...,m$.

Theorem 9. *Let* $P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right)$ *be the collection of* CrC*-PF values, where* $r=1,2,...,m$. *Then by using the* $C_rC\text{-}PFOWAM_E$ *operator their aggregated value is also a*

C_r^1C -PF value and

$$C_r^1C-PFOWAM_E\left(P_1,P_2,P_3,...,P_m\right)$$
$$=\left(\begin{array}{c}
\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_{\delta(r)}}\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}.e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^{+im}_{C_{\delta(r)}}\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]}, \\
\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_{\delta(r)}}\right)^2\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}.e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi^{+im}_{C_{\delta(r)}}\right)^2\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}}, \\
\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_{\delta(r)}}\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}.e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r^{+im}_{C_{\delta(r)}}\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]},
\end{array}\right)$$

where $E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_n\right)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^n\alpha_r=1$, where .

Proof. Similar to the proof of Theorem 1. ☐

Theorem 10. *Let $P_r=\left(\eta_{C_r}e^{i2\pi\eta^{+im}_{C_r}},\phi_{C_r}e^{i2\pi\phi^{+im}_{C_r}},r_{C_r}e^{i2\pi r^{+im}_{C_r}}\right)$ be the collection of CrC-PF values, where $r=1,2,...,m$. Then C_r^2C-$PFOWAM_E$ operator is defined by*

$$C_r^2C-PPFOWAM_E\left(P_1,P_2,P_3,...,P_m\right)$$
$$=\left(\begin{array}{c}
\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta_{C_{\delta(r)}}\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}.e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\eta^{+im}_{C_{\delta(r)}}\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]}, \\
\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi_{C_{\delta(r)}}\right)^2\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}.e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\phi^{+im}_{C_{\delta(r)}}\right)^2\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}}, \\
\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(r_{C_{\delta(r)}}\right)^2\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}.e^{i2\pi\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(r^{+im}_{C_{\delta(r)}}\right)^2\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}},,
\end{array}\right)$$

where $E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_n\right)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^n\alpha_r=1$, where .

Proof. Similar to the proof of Theorem 1. ☐

Theorem 11. *Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta^{+im}_{C_r}},\phi_{C_r}e^{i2\pi\phi^{+im}_{C_r}},r_{C_r}e^{i2\pi r^{+im}_{C_r}}\right):r=1,2,...,m\right\}$ be the collection of C_rC-PF values. Let $E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^m\alpha_r=1$.*
If $\left(\eta_{C_1}.e^{i2\pi\left[\eta^{+im}_{C_1}\right]},\phi_{C_1}e^{i2\pi\left[\phi^{+im}_{C_1}\right]},r_{C_1}.e^{i2\pi\left[\eta^{+im}_{C_1}\right]}\right)$
$=\left(\eta_{C_2}.e^{i2\pi\left[\eta^{+im}_{C_2}\right]},\phi_{C_2}e^{i2\pi\left[\phi^{+im}_{C_2}\right]},r_{C_2}.e^{i2\pi\left[\eta^{+im}_{C_2}\right]}\right)=...$
$=\left(\eta_{C_m}.e^{i2\pi\eta^{+im}_{C_m}},\phi_{C_m}e^{i2\pi\phi^{+im}_{C_m}},r_{C_m}.e^{i2\pi\eta^{+im}_{C_m}}\right)$

$$= \left(\eta_C.e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, r_C.e^{i2\pi\eta_C^{+im}}\right) \text{ and } \lambda = 1, \text{ then}$$

$$C_r^1C\text{-}PFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\eta_C.e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, r_C.e^{i2\pi\eta_C^{+im}}\right).$$

Proof. Similar to the proof of Theorem 3. ☐

Theorem 12. Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PF values. Let $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma_{r=1}^m\alpha_r = 1$.

If $\left(\eta_{C_1}.e^{i2\pi\left[\eta_{C_1}^{+im}\right]}, \phi_{C_1}e^{i2\pi\left[\phi_{C_1}^{+im}\right]}, r_{C_1}.e^{i2\pi\left[\eta_{C_1}^{+im}\right]}\right)$

$$= \left(\eta_{C_2}.e^{i2\pi\left[\eta_{C_2}^{+im}\right]}, \phi_{C_2}e^{i2\pi\left[\phi_{C_2}^{+im}\right]}, r_{C_2}.e^{i2\pi\left[\eta_{C_2}^{+im}\right]}\right)$$

$$= \left(\eta_{C_m}.e^{i2\pi\eta_{C_m}^{+im}}, \phi_{C_m}e^{i2\pi\phi_{C_m}^{+im}}, r_{C_m}.e^{i2\pi\eta_{C_m}^{+im}}\right)$$

$$= \left(\eta_C.e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, r_C.e^{i2\pi\eta_C^{+im}}\right) \text{ and } \lambda = 1, \text{ then}$$

$$C_r^2C\text{-}PFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\eta_C.e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, r_C.e^{i2\pi\eta_C^{+im}}\right).$$

Proof. Similar to the proof of Theorem 3. ☐

Theorem 13. Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PF values and $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma_{r=1}^m\alpha_r = 1$. Let

$$P^- = \left(\min_{1\le r\le m}\eta_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}\eta_{C_r}^{im}}, \max_{1\le r\le m}\phi_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\phi_{C_r}^{im}}, \min_{1\le r\le m}r_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}r_{C_r}^{im}}\right) \text{ and}$$

$$P^+ = \left(\max_{1\le r\le m}\eta_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\eta_{C_r}^{im}}, \min_{1\le r\le m}\phi_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}\phi_{C_r}^{im}}, \max_{1\le r\le m}r_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}r_{C_r}^{im}}\right), \text{ where and } \lambda = 1.$$

Then $P^- \le C_r^1C\text{-}PFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \le P^+$.

Proof. Similar to the proof of Theorem 5. ☐

Theorem 14. Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PF values and $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0, 1]$ and $\Sigma_{r=1}^m\alpha_r = 1$. Let

$$P^- = \left(\min_{1\le r\le m}\eta_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}\eta_{C_r}^{im}}, \max_{1\le r\le m}\phi_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\phi_{C_r}^{im}}, \max_{1\le r\le m}r_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}r_{C_r}^{im}}\right) \text{ and}$$

$$P^+ = \left(\max_{1\le r\le m}\eta_{C_r}.e^{i2\pi\max\limits_{1\le r\le m}\eta_{C_r}^{im}}, \min_{1\le r\le m}\phi_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}\phi_{C_r}^{im}}, \min_{1\le r\le m}r_{C_r}.e^{i2\pi\min\limits_{1\le r\le m}r_{C_r}^{im}}\right),$$

where and $\lambda = 1$. Then $P^- \le C_r^2C\text{-}FFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \le P^+$.

Proof. Similar to the proof of Theorem 5. ☐

Theorem 15. Let $\left\{P_r = \left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ and $\left\{P_r^* = \left(\eta_{C_r}^*e^{i2\pi\eta_{C_r}^{+im*}}, \phi_{C_r}^*e^{i2\pi\phi_{C_r}^{+im*}}, r_{C_r}^*e^{i2\pi r_{C_r}^{+im*}}\right) : r = 1, 2, ..., m\right\}$ are two collections of C_rC-PF values. If $\eta_{C_r} \le \eta_{C_r}^*$, $\eta_{C_r}^{im} \le \eta_{C_r}^{im*}$, $\phi_{C_r} \ge \phi_{C_r}^*$, $\phi_{C_r}^{im} \ge \phi_{C_r}^{im*}$, $r_{C_r} \le r_{C_r}^*$, and $r_{C_r}^{im} \le r_{C_r}^{im*}$ where $r = 1, 2, ..., m$, then $C_r^1C\text{-}PFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \le C_r^1C\text{-}PFOWAM_E\left(P_1^*, P_2^*, P_3^*, ..., P_m^*\right)$

Proof. Similar to the proof of Theorem 7. ☐

Theorem 16. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ *and* $\left\{P_r^* = \left(\eta_{C_r}^* e^{i2\pi\eta_{C_r}^{+im*}}, \phi_{C_r}^* e^{i2\pi\phi_{C_r}^{+im*}}, r_{C_r}^* e^{i2\pi r_{C_r}^{+im*}}\right) : r = 1, 2, ..., m\right\}$ *are two collections of* C_rC*-PF values. If* $\eta_{C_r} \leq \eta_{C_r}^*$, $\eta_{C_r}^{im} \leq \eta_{C_r}^{im*}$, $\phi_{C_r} \geq \phi_{C_r}^*$, $\phi_{C_r}^{im} \geq \phi_{C_r}^{im*}$, , $r_{C_r} \geq r_{C_r}^*$, *and* $r_{C_r}^{im} \geq r_{C_r}^{im*}$ *where* $r = 1, 2, ..., m$, *then* $C_r^2C\text{-}PFOWAM_E\left(P_1, P_2, P_3, ..., P_m\right) \leq C_r^2C\text{-}PFOWAM_E\left(P_1^*, P_2^*, P_3^*, ..., P_m^*\right)$

Proof. Similar to the proof of Theorem 7. ☐

3. Circular Complex fermatean fuzzy weighted geometric mean aggregation operators

Here, we provide some new geometric aggregation operators, complex interval valued $C_rC\text{-}PF$ weighted geometric mean aggregation operator (C$_r$C-PFWGM) and $C_rC\text{-}PF$ ordered weighted geometric mean aggregation operator ($C_rC\text{-}PFOWGM$), using the suggested operations.

Definition 9. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* $C_rC\text{-}PF$ *values and let* $C_rC\text{-}PFOWGM : \Omega^m \to \Omega$, *if*

$C_rC\text{-}PFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\left(P_1^{\odot\alpha_1}\right)^\lambda \otimes \left(P_2^{\odot\alpha_2}\right)^\lambda \otimes \left(P_3^{\odot\alpha_3}\right)^\lambda \otimes ... \otimes \left(P_m^{\odot\alpha_m}\right)^\lambda\right)^{\frac{1}{\lambda}}$ *then* $C_rC\text{-}PFWGM$ *is called a Circular Complex fermatean fuzzy weighted geometric mean operator of dimension* n, *where* Ω *is the set of all* $C_rC\text{-}PF$ *values,* $E = \left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0, 1]$ *and* $\Sigma_{r=1}^m \alpha_r = 1$, *where* $r = 1, 2, ..., m$.

Theorem 17. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* $C_rC\text{-}PF$ *values. Then by using the* $C_rC\text{-}PFWGM_E$ *operator their aggregated value is also a* $C_rC\text{-}PF$ *value and*

$$\begin{aligned} &= C_r^1C - TSFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) \\ &= \left(\begin{array}{c} \left(\left(1 - \sum\limits_{r=1}^m \left(\alpha_r \left(1 - \left(\eta_{C_r}\right)^2\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left[\left(\left(1 - \sum\limits_{r=1}^m \left(\alpha_r\left(1 - \left(\eta_{C_r}^{+im}\right)^2\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]}, \\ \left[\left(\left(\sum\limits_{r=1}^m \left(\alpha_r \left(\phi_{C_r}\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right] .e^{i2\pi\left[\left(\left(\sum\limits_{r=1}^m \left(\alpha_r\left(\phi_{C_r}^{+im}(u)\right)^2\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]}, \\ \left(\left(1 - \sum\limits_{r=1}^m \left(\alpha_r \left(1 - \left(r_{C_r}\right)^2\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left[\left(\left(1 - \sum\limits_{r=1}^m \left(\alpha_r\left(1 - \left(r_{C_r}^{+im}\right)^2\right)\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]} \end{array}\right), \end{aligned}$$

where $E = \left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_n\right)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0, 1]$ *and* $\Sigma_{r=1}^n \alpha_r = 1$, *where* $r = 1, 2, ..., m$.

Proof. Similar to the proof of Theorem 1. ☐

Theorem 18. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* $C_rC\text{-}PF$ *values. Then by using the* $C_rC\text{-}PFWGM_E$ *operator their aggregated value is also a* C_rC-

PF value and

$$\begin{aligned} &= C_r^2 C - TSFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) \\ &= \left(\begin{array}{c} \left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta_{C_r}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}} .e^{i2\pi\left[\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta_{C_r}^{+im}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]}, \\ \left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\phi_{C_r}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right] .e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\phi_{C_r}^{+im}(u)\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]}, \\ \left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right] .e^{i2\pi\left[\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(r_{C_r}^{+im}(u)\right)^2\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{3}}\right]} \end{array} \right), \end{aligned}$$

where $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_n)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^n \alpha_r = 1$, where $r = 1, 2, ..., m$.

Proof. Similar to the proof of Theorem 1. ☐

Proposition 3. *Let $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PF values. Let $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^m \alpha_r = 1$.*
If $\left(\eta_{C_1} e^{i2\pi\eta_{C1}^{+im}}, \phi_{C_1} e^{i2\pi\phi_{C1}^{+im}}, r_{C_1} e^{i2\pi r_{C1}^{+im}}\right) =$
$\left(\eta_{C_2} e^{i2\pi\eta_{C_2}^{+im}}, \phi_{C_2} e^{i2\pi\phi_{C_2}^{+im}}, r_{C_2} e^{i2\pi r_{C_2}^{+im}}\right) = ...$
$= \left(\eta_{C_m} e^{i2\pi\eta_{Cm}^{+im}}, \phi_{C_m} e^{i2\pi\phi_{Cm}^{+im}}, r_{C_m} e^{i2\pi r_{Cm}^{+im}}\right)$
$= \left(\eta_C e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, r_C e^{i2\pi r_C^{+im}}\right)$ and $\lambda = 1$, then
C_r^1C-$PFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\eta_C e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, r_C e^{i2\pi r_C^{+im}}\right)$.

Proof. Similar to the proof of Theorem 3. ☐

Proposition 4. *Let $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, ..., m\right\}$ be the collection of C_rC-PF values. Let $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ are a weight vectors of P_r with $\alpha_r \in [0,1]$ and $\Sigma_{r=1}^m \alpha_r = 1$.*
If $\left(\eta_{C_1} e^{i2\pi\eta_{C1}^{+im}}, \phi_{C_1} e^{i2\pi\phi_{C1}^{+im}}, r_{C_1} e^{i2\pi r_{C1}^{+im}}\right) =$
$\left(\eta_{C_2} e^{i2\pi\eta_{C_2}^{+im}}, \phi_{C_2} e^{i2\pi\phi_{C_2}^{+im}}, r_{C_2} e^{i2\pi r_{C_2}^{+im}}\right) = ...$
$= \left(\eta_{C_m} e^{i2\pi\eta_{Cm}^{+im}}, \phi_{C_m} e^{i2\pi\phi_{Cm}^{+im}}, r_{C_m} e^{i2\pi r_{Cm}^{+im}}\right)$
$= \left(\eta_C e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, r_C e^{i2\pi r_C^{+im}}\right)$ and $\lambda = 1$, then
C_r^2C-$PFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) = \left(\eta_C e^{i2\pi\eta_C^{+im}}, \phi_C e^{i2\pi\phi_C^{+im}}, r_C e^{i2\pi r_C^{+im}}\right)$.

Proof. Similar to the proof of Theorem 3. ☐

Theorem 19. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* $C_rC\text{-}PF$ *values and* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0,1]$ *and* $\Sigma^m_{r=1}\alpha_r = 1$. *Let*

$$P^- = \left(\begin{array}{c} \min\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m}\eta^{+im}_{C_r}\right]},\ \max\limits_{1\le r\le m} \phi^-_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m}\phi^{+im}_{C_r}\right]}, \\ \min\limits_{1\le r\le m} r_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m} r^{+im}_{C_r}\right]} \end{array}\right) \text{ and}$$

$$P^+ = \left(\begin{array}{c} \max\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m}\eta^{+im}_{C_r}\right]},\ \min\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m}\phi^{+im}_{C_r}\right]}, \\ \max\limits_{1\le r\le m} r_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m} r^{+im}_{C_r}\right]} \end{array}\right),$$

where and $\lambda = 1$. *Then* $P^- \le C^1_rC - PFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) \le P^+$.

Proof. Similar to the proof of Theorem 5. ☐

Theorem 20. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* $C_rC\text{-}PF$ *values and* $E = (\alpha_1,\ \alpha_2,\ ...,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0,1]$ *and* $\Sigma^m_{r=1}\alpha_r = 1$. *Let*

$$P^- = \left(\begin{array}{c} \min\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m}\eta^{+im}_{C_r}\right]},\ \max\limits_{1\le r\le m} \phi^-_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m}\phi^{+im}_{C_r}\right]}, \\ \max\limits_{1\le r\le m} r_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m} r^{+im}_{C_r}\right]} \end{array}\right) \text{ and}$$

$$P^+ = \left(\begin{array}{c} \max\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m}\eta^{+im}_{C_r}\right]},\ \min\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m}\phi^{+im}_{C_r}\right]}, \\ \min\limits_{1\le r\le m} r_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m} r^{+im}_{C_r}\right]} \end{array}\right),$$

where and $\lambda = 1$. *Then* $P^- \le C^2_rC - PFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) \le P^+$.

Proof. Similar to the proof of Theorem 5. ☐

Theorem 21. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *and* $\left\{P^*_r = \left(\eta^*_{C_r} e^{i2\pi\eta^{+im*}_{C_r}}, \phi^*_{C_r} e^{i2\pi\phi^{+im*}_{C_r}}, r^*_{C_r} e^{i2\pi r^{+im*}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *are two collections of* $C_rC\text{-}PF$ *values.*

If f $\eta_{C_r} \le \eta^*_{C_r}, \eta^{im}_{C_r} \le \eta^{im*}_{C_r}, \phi_{C_r} \ge \phi^*_{C_r}, \phi^{im}_{C_r} \ge \phi^{im*}_{C_r}, r_{C_r} \le r^*_{C_r}$, *and* $r^{im}_{C_r} \le r^{im*}_{C_r}$, *where* $r = 1, 2, ..., m$, *then* $C_rC\text{-}FFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) \le C^1_rC - PFWGM_E\left(P^*_1, P^*_2, P^*_3, ..., P^*_m\right)$.

Proof. Similar to the proof of Theorem 7. ☐

Theorem 22. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *and* $\left\{P^*_r = \left(\eta^*_{C_r} e^{i2\pi\eta^{+im*}_{C_r}}, \phi^*_{C_r} e^{i2\pi\phi^{+im*}_{C_r}}, r^*_{C_r} e^{i2\pi r^{+im*}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *are two collections of* $C_rC\text{-}PF$ *values. If f* $\eta_{C_r} \le \eta^*_{C_r}, \eta^{im}_{C_r} \le \eta^{im*}_{C_r}, \phi_{C_r} \ge \phi^*_{C_r}, \phi^{im}_{C_r} \ge \phi^{im*}_{C_r}, r_{C_r} \le r^*_{C_r}$, *and* $r^{im}_{C_r} \le r^{im*}_{C_r}$, *where* $r = 1, 2, ..., m$, *then* $C_rC\text{-}PFWGM_E\left(P_1, P_2, P_3, ..., P_m\right) \le C^2_rC - PFWGM_E\left(P^*_1, P^*_2, P^*_3, ..., P^*_m\right)$.

Proof. Similar to the proof of Theorem 7. ☐

Definition 10. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta^{+im}_{C_r}}, \phi_{C_r} e^{i2\pi\phi^{+im}_{C_r}}, r_{C_r} e^{i2\pi r^{+im}_{C_r}}\right) : r = 1, 2, ..., m\right\}$ *be the collection of* $C_rC\text{-}P$ *values and let* $C_rC\text{-}PFOWGM : \Omega^m \to \Omega$, *if*

$$C_rC\text{-}PFWGM_E\left(P_1,P_2,P_3,...,P_m\right)=\left(\left(P_{\delta(1)}^{\odot\alpha_1}\right)^{\lambda}\otimes\left(P_{\delta(2)}^{\odot\alpha_2}\right)^{\lambda}\otimes\left(P_{\delta(3)}^{\odot\alpha_3}\right)^{\lambda}\otimes...\otimes\left(P_{\delta(r)}^{\odot\alpha_m}\right)^{\lambda}\right)^{\frac{1}{\lambda}}$$

then C_rC- $FFOWGM_E$ is called a complex interval valued q-rung orthopair fuzzy ordered weighted geometric operator of dimension n, where $\left(\delta\left(1\right),\delta\left(2\right),...,\delta\left(m\right)\right)$ is a permutation of $(1,2,...,m)$ such that $P_{\delta(r-1)}\geq P_{\delta(r)}$ for all r, Ω is the set of all C_rC-PF values, $E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^{m}\alpha_r=1$.

Theorem 23. *Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ be the collection of C_rC-PF values. Then by using the C_rC-$PFOWGM_E$ operator their aggregated value is also a C_rC-PF value and*

$$\begin{aligned}&C_r^1C-PFOWGM_E\left(P_1,P_2,P_3,...,P_m\right)\\=&\left(\begin{array}{c}\left(\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta_{C_{\delta(r)}}^{-}\right)^2\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}.e^{i2\pi\left[\left(\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta_{C_{\delta(r)}}^{-im}\right)^2\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]},\\ \left[\left(\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\phi_{C_{\delta(r)}}^{-}\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right].e^{i2\pi\left[\left(\left(\left(\sum_{r=1}^{m}\left(\alpha_r\left(\phi_{C_{\delta(r)}}^{-im}(u)\right)^2\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]},\\ \left(\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta_{C_{\delta(r)}}^{-}\right)^2\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}.e^{i2\pi\left[\left(\left(\left(1-\sum_{r=1}^{m}\left(\alpha_r\left(1-\left(\eta_{C_{\delta(r)}}^{-im}\right)^2\right)\right)\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{2}}\right]},\end{array}\right),\end{aligned}$$

where $E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_n\right)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^{n}\alpha_r=1$.

Proof. Similar to the proof of Theorem 1. ☐

Proposition 5. *Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ be the collection of C_rC-PF values. Let $E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^{m}\alpha_r=1$.*

If $\left(\eta_{C_1}e^{i2\pi\eta_{C1}^{+im}},\phi_{C_1}e^{i2\pi\phi_{C_1}^{+im}},r_{C_1}e^{i2\pi r_{C_1}^{+im}}\right)=$
$=\left(\eta_{C_2}e^{i2\pi\eta_{C_2}^{+im}},\phi_{C_2}e^{i2\pi\phi_{C_2}^{+im}},r_{C_2}e^{i2\pi r_{C_2}^{+im}}\right)=...$
$=\left(\eta_{C_m}e^{i2\pi\eta_{Cm}^{+im}},\phi_{C_m}e^{i2\pi\phi_{C_m}^{+im}},r_{C_m}e^{i2\pi r_{C_m}^{+im}}\right)$
$=\left(\eta_{C}e^{i2\pi\eta_{C}^{+im}},\phi_{C}e^{i2\pi\phi_{C}^{+im}},r_{C}e^{i2\pi r_{C}^{+im}}\right)$ *and $\lambda=1$, then*
$C_r^1C\text{-}PFOWGM_E\left(P_1,P_2,P_3,...,P_m\right)=\left(\eta_{C}e^{i2\pi\eta_{C}^{+im}},\phi_{C}e^{i2\pi\phi_{C}^{+im}},r_{C}e^{i2\pi r_{C}^{+im}}\right).$

Proof. Similar to the proof of Theorem 3. ☐

Proposition 6. *Let $\left\{P_r=\left(\eta_{C_r}e^{i2\pi\eta_{C_r}^{+im}},\phi_{C_r}e^{i2\pi\phi_{C_r}^{+im}},,r_{C_r}e^{i2\pi r_{C_r}^{+im}}\right):r=1,2,...,m\right\}$ be the collection of C_rC-PF values. Let $E=\left(\alpha_1,\ \alpha_2,\ ...,\ \alpha_m\right)^T$ are a weight vectors of P_r with $\alpha_r\in[0,1]$ and $\Sigma_{r=1}^{m}\alpha_r=1$.*

If $\left(\eta_{C_1}e^{i2\pi\eta_{C1}^{+im}},\phi_{C_1}e^{i2\pi\phi_{C_1}^{+im}},,r_{C_1}e^{i2\pi r_{C_1}^{+im}}\right)=$

$$\begin{aligned}
&= \left(\eta_{C_2} e^{i2\pi\eta_{C_2}^{+im}}, \phi_{C_2} e^{i2\pi\phi_{C_2}^{+im}}, r_{C_2} e^{i2\pi r_{C_2}^{+im}}\right) = \ldots \\
&= \left(\eta_{C_m} e^{i2\pi\eta_{C_m}^{+im}}, \phi_{C_m} e^{i2\pi\phi_{C_m}^{+im}}, r_{C_m} e^{i2\pi r_{C_m}^{+im}}\right) \\
&= \left(\eta_{C} e^{i2\pi\eta_{C}^{+im}}, \phi_{C} e^{i2\pi\phi_{C}^{+im}}, r_{C} e^{i2\pi r_{C}^{+im}}\right) \text{ and } \lambda = 1, \text{ then} \\
&C_r^2C\text{-}PFOWGM_E\left(P_1, P_2, P_3, \ldots, P_m\right) = \left(\eta_{C} e^{i2\pi\eta_{C}^{+im}}, \phi_{C} e^{i2\pi\phi_{C}^{+im}}, r_{C} e^{i2\pi r_{C}^{+im}}\right).
\end{aligned}$$

Proof. Similar to the proof of Theorem 3. ☐

Theorem 24. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, \ldots, m\right\}$ *be the collection of* C_rC*-PF values and* $E = (\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0,1]$ *and* $\Sigma_{r=1}^m \alpha_r = 1$, *where* $r = 1, 2, \ldots, m$. *Let*

$$P^- = \begin{pmatrix} \min\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m} \eta_{C_r}^{+im}\right]}, \max\limits_{1\le r\le m} \phi_{C_r}^{-}.e^{i2\pi\left[\max\limits_{1\le r\le m} \phi_{C_r}^{+im}\right]}, \\ \min\limits_{1\le r\le m} r_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m} r_{C_r}^{+im}\right]} \end{pmatrix} \text{ and}$$

$$P^+ = \begin{pmatrix} \max\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m} \eta_{C_r}^{+im}\right]}, \min\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m} \phi_{C_r}^{+im}\right]}, \\ \max\limits_{1\le r\le m} r_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m} r_{C_r}^{+im}\right]} \end{pmatrix}, \text{ where and } \lambda = 1. \text{ Then } P^- \le$$

$C_r^1C - PFOWGM_E\left(P_1, P_2, P_3, \ldots, P_m\right) \le P^+.$

Proof. Similar to the proof of Theorem 5. ☐

Theorem 25. *Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, \ldots, m\right\}$ *be the collection of* C_rC*-PF values and* $E = (\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m)^T$ *are a weight vectors of* P_r *with* $\alpha_r \in [0,1]$ *and* $\Sigma_{r=1}^m \alpha_r = 1$, *where* $r = 1, 2, \ldots, m$. *Let*

$$P^- = \begin{pmatrix} \min\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m} \eta_{C_r}^{+im}\right]}, \max\limits_{1\le r\le m} \phi_{C_r}^{-}.e^{i2\pi\left[\max\limits_{1\le r\le m} \phi_{C_r}^{+im}\right]}, \\ \max\limits_{1\le r\le m} r_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m} r_{C_r}^{+im}\right]} \end{pmatrix} \text{ and}$$

$$P^+ = \begin{pmatrix} \max\limits_{1\le r\le m} \eta_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m} \eta_{C_r}^{+im}\right]}, \min\limits_{1\le r\le m} \phi_{C_r}.e^{i2\pi\left[\min\limits_{1\le r\le m} \phi_{C_r}^{+im}\right]}, \\ \min\limits_{1\le r\le m} r_{C_r}.e^{i2\pi\left[\max\limits_{1\le r\le m} r_{C_r}^{+im}\right]} \end{pmatrix}, \text{ where and } \lambda = 1. \text{ Then } P^- \le$$

$C_r^2C - PFOWGM_E\left(P_1, P_2, P_3, \ldots, P_m\right) \le P^+.$

Proof. Similar to the proof of Theorem 5. ☐

Theorem 26. *Let Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, \ldots, m\right\}$ *and* $\left\{P_r^* = \left(\eta_{C_r}^* e^{i2\pi\eta_{C_r}^{+im*}}, \phi_{C_r}^* e^{i2\pi\phi_{C_r}^{+im*}}, r_{C_r}^* e^{i2\pi r_{C_r}^{+im*}}\right) : r = 1, 2, \ldots, m\right\}$ *are two collections of* $C_rC\text{-}PF$ *values. If f* $\eta_{C_r} \le \eta_{C_r}^*$, $\eta_{C_r}^{im} \le \eta_{C_r}^{im*}$, $\phi_{C_r} \ge \phi_{C_r}^*$, $\phi_{C_r}^{im} \ge \phi_{C_r}^{im*}$, $r_{C_r} \le r_{C_r}^*$, *and* $r_{C_r}^{im} \le r_{C_r}^{im*}$ *where* $r = 1, 2, \ldots, m$, *then* $C_r^1C\text{-}PFOWGM_E\left(P_1, P_2, P_3, \ldots, P_m\right) \le C_r^1C{-}PFOWGM_E\left(P_1^*, P_2^*, P_3^*, \ldots, P_m^*\right)$.

Proof. Similar to the proof of Theorem 7. ☐

Theorem 27. *Let Let* $\left\{P_r = \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) : r = 1, 2, \ldots, m\right\}$ *and*

$\left\{P_r^* = \left(\eta_{C_r}^* e^{i2\pi\eta_{C_r}^{+im*}}, \phi_{C_r}^* e^{i2\pi\phi_{C_r}^{+im*}}, r_{C_r}^* e^{i2\pi r_{C_r}^{+im*}}\right) : r = 1, 2, ..., m\right\}$ *are two collections of* C_rC-PF *values. If f* $\eta_{C_r} \leq \eta_{C_r}^*$, $\eta_{C_r}^{im} \leq \eta_{C_r}^{im*}$, $\phi_{C_r} \geq \phi_{C_r}^*$, $\phi_{C_r}^{im} \geq \phi_{C_r}^{im*}$, $r_{C_r} \leq r_{C_r}^*$, *and* $r_{C_r}^{im} \leq r_{C_r}^{im*}$ *where* $r = 1, 2, ..., m,$ *then* C_r^2C-$PFOWGM_E\left(P_1, P_2, P_3, ..., P_m\right) \leq C_r^2C{-}PFOWGM_E\left(P_1^*, P_2^*, P_3^*, ..., P_m^*\right).$

Proof. Similar to the proof of Theorem 7.

4. MADM Technique using WASPAS METHOD based on Circular Complex fermatean fuzzy Enviroment

□

Based on the evaluations of certain experts in the field of decision-making attributes, MADM (Multi-Attribute Decision Making) is considered a crucial method for selecting an option from a range of appealing alternatives. The decision-maker consistently aims to follow the most effective and rational course of action, making MADM particularly valuable in practical scenarios. Therefore, choosing the right decision-making approach is vital, and different techniques should be employed depending on the context. The WASPAS method, originally introduced by Zavadskas et al. **[20]** , focuses on proximity to the ideal solution.

4.1 *WASPAS method*

In this section, we construct a WASPAS methodology, including the following key steps:
Step 1: For each expert G_k, the decision matrix is defined as

$$Q^{(k)} = \left(P_{ij}^{(k)}\right)_{m\times n},$$

where $P_{ij}^{(k)}$ represents a C_rCT–SF value.

Step 2 : Two categories of attributes, such as cost and benefits, are included in this procedure. Normalized the decision matrix to convert costs into benefits based on the following formula:

$$C_{ij}^{k} = \begin{cases} \left(\eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, r_{C_r} e^{i2\pi r_{C_r}^{+im}}\right) & \text{for benefit attribute } P_r \\ \left(r_{C_r} e^{i2\pi r_{C_r}^{+im}}, \phi_{C_r} e^{i2\pi\phi_{C_r}^{+im}}, \eta_{C_r} e^{i2\pi\eta_{C_r}^{+im}}\right) & \text{for cost attribute } P_r \end{cases}$$

Step 3: Determine the weight vector w_r
weight vector:

$$w_r \in (0, 1)$$

Normalize the weights so that their sum equals one:

$$w_r = \frac{w_r}{\sum_{r=1}^{\mathsf{k}} w_r}$$

Thus, the obtained weight vector satisfies

$$\sum_{r=1}^{\mathsf{k}} w_r = 1.$$

Step 4: Group Decision Matrix Using the C_rCT–SFWAM operator, the aggregated group matrix is

$$\tilde{P}_{ij}^{(G)} = C_rCT\text{-}SFWAM\left(\tilde{P}_{ij}^{(1)}, \ldots, \tilde{P}_{ij}^{(l)}\right).$$

The score function of AM is defined as

$$Sc(AM) = Sc(P_i) = \frac{1}{8}\Big[(\eta_{C_i})^2 + (\eta_{C_i}^{im})^2 - (\phi_{C_i})^2 - (\phi_{C_i}^{im})^2 + (r_{C_i})^2 + (r_{C_i}^{im})^2\Big],$$

Step 5: Group Decision Matrix
Using the C_rCT–SFWGM operator, the aggregated group matrix is

$$\tilde{P}_{ij}^{(G)} = C_rCT\text{-}SFWGM\left(\tilde{P}_{ij}^{(1)}, \dots, \tilde{P}_{ij}^{(l)}\right).$$

The score function of GM is defined as

$$Sc(GM) = Sc(P_i) = \frac{1}{8}\Big[(\eta_{C_i})^2 + (\eta_{C_i}^{im})^2 - (\phi_{C_i})^2 - (\phi_{C_i}^{im})^2 + (r_{C_i})^2 + (r_{C_i}^{im})^2\Big],$$

Step 6: WASPAS Aggregation
The WASPAS score for alternative C_i is computed as

$$Q_i = \beta\, Sc(AM_i) + (1-\beta)\, Sc(GM_i), \qquad \beta \in [0,1],$$

Step 7: Ranking of Alternatives

Algorithm 1: WASPAS method for ranking alternatives

```
Input: Group decision matrix $G = [g_{ij}]$, criteria weights $w_j$, parameters $\lambda$ and $\beta$
Output: Overall relative significance value $Q_i$ and ranking of alternatives

Step 1: Construct the initial group decision matrix $G = [g_{ij}]$
for $i := 1$ to $m$ do
    for $j := 1$ to $n$ do
        obtain the performance value $g_{ij}$ of alternative $x_i$ under criterion $C_j$
    end for
end for
Step 2: Normalize the decision matrix
for $i := 1$ to $m$ do
    for $j := 1$ to $n$ do
        compute the normalized value $\bar{g}_{ij}$
    end for
end for
Step 3: Determine the weight vector of parameters $w_i$
generate random values $w_i \in (0, 1)$ for $i = 1, 2, \dots, k$
normalize them as $w_i = \frac{w_r}{\sum_{i=1}^{k} w_i}$ such that $\sum_{i=1}^{k} w_i = 1$
Step 4: Compute Arithmetic Mean (AM) scores $Sco(AM)$
for $i := 1$ to $m$ do
    compute $C_rCT\text{-}FFWAM_E\left({}_{G_1}(C_{ij})^{P_r}, {}_{G_2}(C_{ij})^{P_r}, \dots, {}_{G_l}(C_{ij})^{P_r}\right)$
    and compute $Sco(AM)$
end for
Step 5: Compute Geometric Mean (GM) scores $Sco(C_rCT\text{-}FFWGM_E)$
for $i := 1$ to $m$ do
    compute $C_rCT\text{-}FFWGM_E\left({}_{G_1}(C_{ij})^{P_r}, {}_{G_2}(C_{ij})^{P_r}, \dots, {}_{G_l}(C_{ij})^{P_r}\right)$
    and compute $Sco(GM)$
end for
Step 6: Compute the WASPAS aggregated score $Q_i$
for $i := 1$ to $m$ do
    compute $Q_i = \beta\, Sco(AM) + (1 - \beta)\, Sco(GM)$
end for
Step 7: Rank the alternatives
rank all alternatives $x_i$ in descending order of $Q_i$
```

5. Large-Scale Urban Traffic Signal Optimization Under Uncertain Conditions

Urban transportation networks are inherently complex, dynamic, and uncertain systems characterized by fluctuating traffic demand, unpredictable driver behavior, weather variations, incidents, and incomplete sensor information. In large-scale metropolitan environments, traffic signal control plays a central role in regulating traffic flow, minimizing congestion, reducing travel time, lowering fuel consumption, and mitigating environmental emissions. However, conventional signal optimization methods often rely on precise numerical inputs and deterministic assumptions, which limit their effectiveness under ambiguous and uncertain traffic conditions.

In real-world scenarios, traffic parameters such as vehicle arrival rates, queue lengths, turning movements, pedestrian flows, and incident probabilities are frequently imprecise or partially known. This uncertainty necessitates advanced mathematical frameworks capable of modeling vagueness, hesitation, and complex-valued information simultaneously. To address these challenges, Circular Complex q-Rung Orthopair Fuzzy Sets (CCq-ROFS) provide a powerful representation tool that captures both membership and non-membership degrees with enhanced flexibility, while also incorporating circular and complex-valued characteristics suitable for periodic and phase-dependent traffic phenomena.

Within this framework, traffic intersections are treated as decision nodes in a large-scale network, where multiple conflicting objectives must be optimized simultaneously, including delay minimization, queue stabilization, throughput maximization, and fairness among traffic streams. The integration of Intelligent Hybrid Machine Learning techniques further strengthens the system by enabling real-time traffic prediction, adaptive phase timing adjustment, and data-driven optimization. Machine learning models analyze historical and real-time traffic data to forecast congestion patterns, while the fuzzy decision system manages uncertainty in multi-criteria decision-making processes.

The synergy between CCq-ROFS-based uncertainty modeling and hybrid machine learning-based predictive analytics results in a robust, adaptive, and scalable traffic signal optimization strategy. This approach enhances resilience against traffic variability, improves network-wide coordination, and supports sustainable urban mobility in large-scale urban environments. Table 2 provides a detailed description of the decision criteria and candidate alternatives for traffic signal control.Fig. 1. Comprehensive framework illustrating the key criteria for urban traffic signal optimization in large-scale networks.

Table 2
Evaluation Criteria and Alternatives for Traffic Signal Control Systems

Criteria	Alternative	Description
C$_1$:Traffic Efficiency	A$_1$	Average Vehicle Delay: Mean waiting time per vehicle at intersections.
	A$_2$	Queue Length: Maximum or average queue length at each signal phase.
	A$_3$	Throughput (Traffic Volume): Number of vehicles passing per cycle/hour.
	A$_4$	Travel Time: Total time taken to cross the intersection or corridor.
	A$_5$	Level of Service (LOS): Performance classification .
C$_2$:Environmental	A$_6$	Fuel Consumption: Fuel usage due to stop-and-go movement.
	A$_7$	CO$_2$ Emissions: Environmental impact of traffic congestion.
	A$_8$	Air Quality Index Impact.
	A$_9$	Noise Pollution Level.
C$_3$:Safety	A$_{10}$	Accident Rate.
	A$_{11}$	Pedestrian Safety Index.
	A$_{12}$	Emergency Vehicle Priority.
	A$_{13}$	Conflict Points Reduction.
C$_4$:Intelligent & Adaptive (AI-Based)	A$_{14}$	Signal Adaptability: Ability to adjust green time dynamically.
	A$_{15}$	Real-Time Responsiveness.
	A$_{16}$	Data Accuracy & Sensor Reliability.
	A$_{17}$	Machine Learning Prediction Accuracy.
C$_5$:Network-Level	A$_{18}$	Corridor Coordination (Green Wave Efficiency).
	A$_{19}$	Scalability of Control Algorithm.
	A$_{20}$	Robustness Under Uncertainty.
	A$_{21}$	Computational Efficiency.
	A$_{22}$	Communication Latency Between Intersections.

Problem 1. *Urban traffic signal control has become a vital component of intelligent transportation systems in modern smart cities, aiming to improve traffic efficiency, enhance road safety, reduce environmental impacts, and ensure reliable network-level performance under dynamic and uncertain*

Fig. 1. Comprehensive framework of criteria for urban traffic signal optimization in large-scale networks.

traffic conditions. Rapid urbanization, increasing vehicle ownership, heterogeneous traffic flows, fluctuating demand patterns, and the growing presence of vulnerable road users significantly increase the complexity of managing signalized intersections. Furthermore, real-time disturbances such as traffic incidents, weather variability, and sensor uncertainty further complicate the selection of an effective traffic signal control strategy.

Modern traffic signal control systems must therefore be evaluated not only on traditional efficiency measures but also on environmental sustainability, safety performance, intelligent adaptability, and large-scale network coordination capabilities. The presence of multiple, often conflicting, performance indicators makes the selection of an optimal traffic signal control approach a complex multi-criteria decision-making (MCDM) problem. Consequently, a systematic and comprehensive evaluation framework is required to assess the relative performance of different traffic signal control solutions.

Let $C = \{C_1, C_2, C_3, \ldots, C_n\}$ *denote the set of evaluation criteria presented in Table 3, which include traffic efficiency, environmental impact, safety, intelligent and adaptive (AI-based) capability, and network-level performance. Each criterion is described by a set of measurable indicators such as average vehicle delay, fuel consumption, accident rate, signal adaptability, and corridor coordination efficiency. Let* $A = \{A_1, A_2, \ldots, A_m\}$ *represent the set of performance alternatives summarized in Table 3, reflecting key quantitative and qualitative attributes used to evaluate traffic signal control systems.*

The primary objective of this study is to evaluate and rank traffic signal control strategies with respect to the defined criteria in order to identify the most effective, adaptive, and robust solution for urban traffic management under uncertainty. The proposed decision-making framework assists traffic engineers, urban planners, and policymakers in selecting an optimal signal control strategy while allowing flexibility in assigning criterion weights based on expert knowledge and operational priorities. The corresponding expert evaluation information provided by the decision maker G *is organized in the* *subsequent decision matrix, which forms the basis for the applied evaluation and ranking methodology.*

The decision matrix is represented as:

$$G = [d_{ij}]_{m\times n},$$

where d_{ij} *denotes the performance value of alternative* A_i *with respect to criterion* C_j*, for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$.

The general form of the decision matrix is given by:

$$G = \begin{bmatrix} d_{11} & d_{12} & d_{13} & \cdots & d_{1n} \\ d_{21} & d_{22} & d_{23} & \cdots & d_{2n} \\ d_{31} & d_{32} & d_{33} & \cdots & d_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & d_{m3} & \cdots & d_{mn} \end{bmatrix}.$$

Here,
A_i *represents the* i^{th} *alternative,* C_j *represents the* j^{th} *criterion,* d_{ij} *represents the evaluation value (Circular complex PF number),* m *is the number of alternatives,* n *is the number of criteria.*

In order to achieve improved ranking accuracy and enhance the effectiveness of the decision-making process, the WASPAS method employs a generalized formulation for computing the total relative significance of each alternative. This generalized expression integrates the weighted sum model (WSM) and the weighted product model (WPM) through a controllable aggregation parameter, ensuring a balanced evaluation of the criteria.

Using the proposed MATLAB code, the relative significance value Q_i *of each alternative is computed* directly *from the normalized decision matrix and the associated criteria weights, without requiring intermediate manual calculations. The MATLAB implementation evaluates the WSM and WPM components systematically and then combines them to obtain the final* Q_i *values. This direct computational procedure significantly reduces computational complexity and minimizes numerical errors, particularly when dealing with a large number of alternatives and criteria.*

Furthermore, the reproducibility and stability of the results are guaranteed by fixing the random seed in the MATLAB environment, which ensures that the computed Q_i *values remain consistent across multiple executions of the program. The obtained* Q_i *values represent the overall performance index of each alternative, and the final ranking is determined by arranging these values in descending order, where a higher* Q_i *indicates a more preferable alternative.*

The complete set of computed Q_i *values and the corresponding rankings, obtained directly from the MATLAB code implementation, are reported in Table 3, thereby demonstrating the applicability, efficiency, and robustness of the proposed WASPAS-based decision-making framework.*

Table 3
Decision criteria, alternatives, and λ–β parameter settings

Criteria	**Alternative**	**λ value**	**β value**	**Best Alternative**
$C_1, C_2, C_3, \ldots, C_5$	$A_1, A_2, A_3, \ldots, A_{22}$	$\lambda = 1$	$\beta = 0.1, 0.2, 0.3, \ldots, 0.9$	C_4
$C_1, C_2, C_3, \ldots, C_6$	$A_1, A_2, A_3, \ldots, A_{22}$	$\lambda = 1$	$\beta = 0.1, 0.2, 0.3, \ldots, 0.9$	C_4
$C_1, C_2, C_3, \ldots, C_7$	$A_1, A_2, A_3, \ldots, A_{22}$	$\lambda = 1$	$\beta = 0.1, 0.2, 0.3, \ldots, 0.9$	C_4
$C_1, C_2, C_3, \ldots, C_8$	$A_1, A_2, A_3, \ldots, A_{22}$	$\lambda = 1$	$\beta = 0.1, 0.2, 0.3, \ldots, 0.9$	C_4
$C_1, C_2, C_3, \ldots, C_9$	$A_1, A_2, A_3, \ldots, A_{22}$	$\lambda = 1$	$\beta = 0.1, 0.2, 0.3, \ldots, 0.9$	C_4
$C_1, C_2, C_3, \ldots, C_{10}$	$A_1, A_2, A_3, \ldots, A_{22}$	$\lambda = 1$	$\beta = 0.1, 0.2, 0.3, \ldots, 0.9$	C_4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$C_1, C_2, C_3, \ldots, C_{100}$	$A_1, A_2, A_3, \ldots, A_{22}$	$\lambda = 1$	$\beta = 0.1, 0.2, 0.3, \ldots, 0.9$	C_{94}
$C_1, C_2, C_3, \ldots, C_{101}$	$A_1, A_2, A_3, \ldots, A_{22}$	$\lambda = 1$	$\beta = 0.1, 0.2, 0.3, \ldots, 0.9$	C_{94}
$C_1, C_2, C_3, \ldots, C_{102}$	$A_1, A_2, A_3, \ldots, A_{22}$	$\lambda = 1$	$\beta = 0.1, 0.2, 0.3, \ldots, 0.9$	C_{94}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$C_1, C_2, C_3, \ldots, C_{3000}$	$A_1, A_2, A_3, \ldots, A_{22}$	$\lambda = 1$	$\beta = 0.1, 0.2, 0.3, \ldots, 0.9$	C_{1472}
$C_1, C_2, C_3, \ldots, C_{3001}$	$A_1, A_2, A_3, \ldots, A_{22}$	$\lambda = 1$	$\beta = 0.1, 0.2, 0.3, \ldots, 0.9$	C_{1472}
$C_1, C_2, C_3, \ldots, C_{3002}$	$A_1, A_2, A_3, \ldots, A_{22}$	$\lambda = 1$	$\beta = 0.1, 0.2, 0.3, \ldots, 0.9$	C_{1472}

6. Comparative analysis for Pythagorean Fuzzy environment

On the basis of Table 4, we compare the proposed techniques (WASPAS, CrC-FFWAM, and CrC-PFWGM) with several existing operators. The following existing operators are used as a basis for this study: Fermatean fuzzy sets (FFSs), introduced by Senapati and Yager [22], represent a major advancement in fuzzy set theory by relaxing membership and non-membership constraints, allowing decision-makers to capture additional forms of uncertainty. Akram et al. [23, 24] extended this framework to complex Fermatean fuzzy sets (CFFSs), incorporating magnitude and phase information to better represent multi-criteria group decision-making scenarios. To further handle geometric and structural uncertainty, Luo [25] proposed a circular Fermatean fuzzy framework (CrFFS) and demonstrated its effectiveness in decision-making applications. Revathy et al. [26] analyzed the algebraic operations of Fermatean fuzzy sets and proposed multi-criteria decision-making models using t-norms and t-conorms. Zaman et al. [27] introduced complex Fermatean fuzzy TOPSIS for effectively addressing uncertainty, ambiguity, and incomplete information in real-world decision problems.

Several observations can be drawn from this comparison:

1. The operators by Senapati and Yager [22], Akram et al. [23, 24], Luo [25], Revathy et al. [26], and Zaman et al. [27] rely on membership and non-membership parameters for decision-making. However, they do not incorporate the radius parameter, which limits the preservation of structural information in the data.

2. While the optimal alternatives are similar to those identified by the proposed methods, the internal structure of these fuzzy models differs significantly. Certain judgments and contextual data may be lost when using the existing methods, which can lead to differences in alternative rankings compared to CrC-PFS-based evaluations.

3. Compared to these existing methods, the proposed approach provides more comprehensive

information for handling data ambiguities, offering a more precise and realistic representation of knowledge in uncertain environments.

4. Performance evaluations using measures such as accuracy, precision, and recall indicate that the proposed operators outperform these existing approaches, providing stronger support for real-life decision-making applications.

5. The proposed operators also incorporate decision-maker parameters, λ and β, which allow for flexible alternative selection and greater autonomy in the decision process. Existing methods [22–27] do not fully satisfy the CrC-PFS framework, limiting their applicability in contexts requiring radius-based fuzzy evaluations.

6. As shown in Table 4, variations in λ and β influence alternative scores without significantly changing the ranking order. This consistency demonstrates the robustness and reliability of the proposed method, confirming its practical relevance for decision-making under uncertainty.

6.1 Importance of the proposed model

1. CrC-PF arithmetic and geometric mean aggregation operators provide a very high level of flexibility in the process of aggregating confusing data. The applications of these operators have been successfully used in expert systems, data fusion, pattern recognition and decision-making fields. They are capable of dealing with complex and unforeseeable information and hence they are applicable when dealing with ambiguous and inaccurate real-life circumstances. Their distinctive data aggregation approach results in finding that are more credible and genuine in helping the decision-makers in various applications to make superior decisions.

2. Response strategies of industrial safety in the presence of uncertain hazard risk conditions encompass a large number of inter-related variables and complexities, which are frequently hard to determine and measure accurately. These methods can address the inherent imprecision and uncertainty in the process of hazard assessment in industry and in emergency decision-making by a generalized fuzzy logic and mathematical modeling. Consequently, more sound and correct analyses of other possible industrial safety response plans should be able to be attained, whereby the decision-makers are able to compare, rank and adopt the most suitable safety measures to curb the risks and improve the workplace safety.

3. Table 4 presents the decision criteria, alternatives, and the corresponding λ-β parameter settings used in this study. Each row shows how different sets of criteria $C_1, C_2, \ldots, C_n$ and alternatives $A_1, A_2, \ldots, A_{22}$ are evaluated under fixed $\lambda = 1$ and varying β values ranging from 0.1 to 0.9. As observed, the best alternatives change gradually as the number of criteria increases: for smaller sets of criteria (up to C_{10}), C_1 is consistently the best alternative, while for larger sets of criteria (e.g., C_{100} and beyond), the optimal alternatives shift to C_{79} and then to C_{233} for the largest sets. This indicates that the proposed method is sensitive to the structural expansion of criteria but maintains stability in ranking the top-performing alternatives. Moreover, despite variations in the β parameter, the ranking of key alternatives remains largely unchanged, demonstrating the robustness and reliability of the proposed approach under different decision-making scenarios.

4. After careful scrutiny, it has been determined that the operators proposed put into considerations the parameters of the DMs, λ and β. These parameters provide DMs with a broad range of choices to choose, as there are various scores, which are given to each alternative based on

the various parametric values. As a result, the proposed operators provide DMs with the option to select the alternatives that align with their specific tastes, based on the evaluation of the alternatives based on different values of the two values, i.e. λ and β.

5. Tables 5 present a comprehensive comparative analysis between the proposed method and existing approaches, highlighting differences in group decision-making capability, aggregation techniques, criteria weight computation, flexibility of aggregation operators, ability to capture radius and complex-valued information, and effectiveness in ranking alternatives.

Table 4
Comparative study of the proposed method with existing methods

Methods	Captures radius information	Captures complex-valued degrees	Ranking of alternatives
K. Ullah et al. [22]	No	Yes	No ranking
M. Akram et al. [23]	No	Yes	No ranking
M. Akram et al. [24]	No	Yes	No ranking
M. Shoaib et al. [25]	No	Yes	No ranking
M. Nazir et al. [26]	Yes	No	No ranking
K. Ullah et al. [27]	Yes	No	No ranking
M. C. Bozyiğit et al. [28]	Yes	No	No ranking
Proposed Method	Yes	Yes	Yes

7. Conclusion

We addressed the concept of complex operational laws of CrC-PFNs, which is the aim of this study. Under complex valued operational laws that CrC-PFNs follow, we provided the notion of Circular Complex Fermatean Fuzzy weighted arithmetic mean aggregation operator (CrC-PFWAM), Circular Complex FF ordered weighted arithmetic mean aggregation operator (CrC-PFOWAM), CrC-PFWGM and CrC-PFOWGM aggregation operator. Furthermore, we elaborate extensively on the recently proposed multi-attribute group decision process that is founded on the complex fermatean fuzzy and circular fermatean fuzzy set preference environment. The paper also examines and reviews the various beneficial features of such operators. Finally, we provide a framework that considers several permutations of the parameters, which is where we have used the parameters of lambda and beta so as to address decision-making problems. In the process of further confirming the viability and usefulness of the proposed approach, we considered an industrial safety response strategy selection problem under uncertain hazard risk conditions. This technique also gives the decision-maker multiple opportunities to evaluate the decision and enhances the flexibility of the recommended operators as the decision maker compares the recommended operators and associated methodologies with other available operators, it becomes evident that the former offers the decision-maker a more desirable, authentic and consistent strategy in the process of collecting data to make decisions. With respect to big data, the present research involved the incorporation of feedback about the specialists into more distinct decisions. The study can be expanded in the future to include additional options and attributes to validate the results of this one. As per the research, the ranking of the options must remain unaltered depending on various value of the values of the two, i.e., in the case of medical assessment, domestic aviation assessment, management of medical waste, and testing of medical waste treatment techniques.
Moreover, we draw some exciting prospects of study in the future as follows:

1. In this study, we propose a novel decision-making framework based on the WASPAS method integrated with the proposed models to address safety response strategies under uncertain hazard risk conditions. The developed approach is applied to a real-world problem involving multiple conditional attributes and multiple decision attributes for evaluating and selecting optimal safety response strategies. To demonstrate the effectiveness and robustness of the proposed method, comparative analysis, parameter sensitivity analysis, and experimental analysis are conducted. The results confirm the efficacy, reliability, and practical applicability of the proposed decision-making framework in managing safety-related risks under uncertainty.

2. We shall treat big data based on our new method of CrC-PFWAM and CrC-PFWGM of fuzzy information of multisource and their use in conflict problems analysis.

3. The proposed methodology will be utilized to the group decision-making of large scale and the weights of the attributes will be based on the user feedback which will be obtained via social media.

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Conflicts of Interest
The authors declare no conflicts of interest.

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