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Complex Pythagorean Fuzzy Aczel–Alsina Weighted Heronian Mean Operators and Their Applications in MCDM

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ABSTRACT

Aggregation operators serve as fundamental mathematical tools for synthesizing multiple inputs into a single representative output. In this study, we introduce a family of aggregation operators (AOs) tailored to the framework of complex Pythagorean fuzzy sets (CPyFSs). Owing to their superior ability to capture both amplitude and phase information, CPyFSs offer enhanced flexibility in modeling uncertainty and vagueness, making them highly suitable for real-world decision-making problems. Motivated by these advantages, this paper develops several novel AOs within the CPyFS environment to address multi-attribute decision-making (MADM) problems more effectively. To further enhance the flexibility and robustness of the proposed operators, Aczél–Alsina (AA) operational laws are incorporated. In particular, we propose the complex Pythagorean fuzzy Aczél–Alsina Heronian mean (CPyFAAHM) operator and the complex Pythagorean fuzzy Aczél–Alsina geometric Heronian mean (CPyFAAGHM) operator, which integrate AA operations with Heronian mean and geometric Heronian mean structures. We investigate the key mathematical properties of the proposed operators, including their structural characteristics and aggregation behavior. Additionally, weighted versions of these operators are developed to accommodate the varying importance of decision attributes. Based on these operators, a MADM methodology is constructed for decision-making under the CPyFS framework. Finally, the effectiveness and practicality of the proposed approach are demonstrated through a numerical example, followed by a comparative analysis with existing MADM methods under complex Pythagorean fuzzy information, highlighting its superiority and applicability.

1. Introduction

In 1965, Zadeh [1] imported new intellection of fuzzy sets (FSs) by introducing a membership value that ranges from 0 to 1. Zadeh consecration of FSs was a decision-making platform. Later, Atanassov [2] improved the fuzzy intellection by creating intuitionistic fuzzy sets (IFSs), in that

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scenario, the combined value of membership (MV) coupled with non-membership (NMV) is in interval [0, 1]. Yager [3] proposes a new concept of IFSs in the form of Pythagorean fuzzy sets (PyFs). With this concept, IFSs is no longer used to manage ambiguous information such as the totality of the multiplying a number by itself of MV along with NMV in the interval [0, 1], that is, $0 \leq \pi^2 + \Xi^2 \leq 1$. In order to encode the ambiguous information within the medical diagnosis system, Adlassnig [4] modified the idea of FSs.

Numerous studies have provided many examples of the above fuzzy environments. Furthermore, Atanassov [5] analyzed his IFSs concept in context of interval-valued IFSs (IVIFSs) analysis includes both uppercase and lowercase MV and NMV. Mohd and Abdullah [6] propose an analysis of various similarly measured distances using PyFSs. The multi-criteria decision-making (MCDM) problem has been solved through continued research in an expanded set that includes FSs, IFSs, and PyFSs configurations. These imitations are limited to addressing vagueness and ambiguity in the data and cannot accentuate the ignorance of commemorated cognition or dossier fragility. Notwithstanding of, a multifarious information assortment has capacity to handle both the periodicity and irresolution of the information at the same time. Ramot *et al.,* [7, 8] propose the complex fuzzy sets (CFSs) to address these challenges, in which the space could be a unit circle. Complex intuitionistic fuzzy sets (CIFSs) was created as an expansion of the hypothesis of CFSs by Alkouri and Salleh [9]. Ullah *et al.,* [10] proposed complex pythagorean fuzzy sets (CPyFSs) as a device for upgrading the productivity in data handling and alter the environment of CIFSs with the multiplying a number by itself of amplitude coupled with phase terms of MV along with NMV separately. To tackle ambiguity and uncertainty in fuzzy systems, Hashmi [11] introduced a new extension of FSs known as Linear Diophantine FSs. Moreover, PyFSs are incapable of managing with circumstances in which the overall of squares of MV and NMV, surpasses one. To bargain with such sort of situation Yager [12] displayed the concept of q-Rung Orthopair FSs (q-ROFSs). In the literary texts, sundry aggregation operators (AOs) grounded on t-norm (T-N) along with t-conorm (T-CN) have been created to aggregate evaluation knowledge in fuzzy environments. Shea *et al.,* [13] come up with fuzzy AOs that can be intuitively derived using Archimedean T-N as well as T-CN. Akram along with Naz [14] proposed his innovative PyFSs idea in his CPyFSs framework where he uses the MADM approach to handle uncertain information. Ali *et al.,* [15] studied the complex q-ROFSs concept and come into being a new-fangled algorithm to address present-world challenges put to use MADM technology. Mahmood [16] developed a bipolar soft set to deal with hazy and unclear data. We looked at a few Xu [17] developed AOs in the form of weighted average operators as well as a few unique situations using IFSs. Wei [18] created several AOs, including ordered WGOs, hybrid WGOs, and induced weighted geometric operators. Peng and Yuan [19] studied certain differences and developed various Pythagorean fuzzy value (PyFV) algorithms using a generalized weighted average operator. Akram *et al.,* [20] examined interval-valued T-fuzzy sets using the Bonferroni mean AOs and further proposed a MADM method applied to problems related to the solar system. The principles of spherical FSs (SFSs) were expanded upon and novel AOs of SFSs based on Dombi aggregation techniques have been provided by Khan *et al.,* [21]. Rahman *et al.,* [22] offered his new AO of PyFS-like weighted average and weighted geometric operators worthy of fundamental properties. Mahmood *et al.,* [23] demonstrated a novel AO, a new concept in PyFSs, incorporated into CPyFSs to counter the sway of ambiguous along with unclear facts on CPyFSs systems. Liu *et al.,* [24] explored the concept of IFS to handle ambiguous information and developed several new aggregation operators by employing the concept of Maclaurin symmetric mean (MSM) operators. Ullah [25] created a list of new AOs based on picture FSs (PFSs) employing the ideas of MSM operators. In addition to discussing their MADM approach, Akram *et al.,* [26] revealed a new PyFS algorithm for selection procedures in the textile sector. Chen [27] introduced new methods by employing the novel concept of prioritized aggregation operators based on IVIFSs. Utilizing the ideas of Archimedean Bonferroni tools under the framework of q-ROFSs, Liu and Wang [28] offered a list of AOs to address a real-life problem based on MADM approaches. Hussain *et al.,* [29] investigated several new aggregation operators and illustrated their applicability by presenting a numerical example to present a MADM problem for selecting suitable tourist destinations. Based on IF data, Garg [30] created a number of novel AOs with entropy weight vectors. Working on a novel idea for Yager AOs, Akram and Shahzadi [31] provided a MADM approach under the q-ROFS system.

Jan *et al.,* [32] developed several additional aggregation operators within the framework of linguistic cubic information to address MADM problems. In order to address a MADM problem, Tahreem *et al.,* [33] laid out various AOs of Complex interval value PyFSs (CIVPyFSs) and used their principles. A novel notion of triangle norms based on probabilistic metric space was put forth by Menger [34] in 1942. Klement [35] used the theories of t-norm (TNM) and t-conorm (TCNM) in various fuzzy information to provide some new aggregation tools. Mahmood *et al.,* [36] devised several aggregation operators based on the Frank t-norm and t-conorm within the framework of interval-valued picture FSs, offering a novel approach to deal with erratic and ambiguous information. Liu [37] investigated the algebraic and Einstein AOs using Hamacher AOs that are dependent on IVIF data. Garg [38] developed a new method for geometric AO by developing the PyFS concept and utilizing Einstein's TNM and TCNM functions in the context of PyFSs. Aczel and Alsina [39], in 1982 propound more reliable and versatile TNM and we have evolved TCNM. In order to choose the finest TNM and TCNM from a family of TNMs and TCNMs, Babu and Ahmed [40] worked on a number of TNMs and TCNMs. Following analysis, they concluded that AA-TNM and AA-TCNM are superior aggregation techniques. In order to get insight into the merits of AA-TNM and AA-TCNM, a number of researchers employed these aggregation methods in their investigations. Senapati *et al.,* [41] examined the ideas of AA-TNM and AA-TCNM and provided an example of how to solve a MAMD approach within an IFS system. Additionally, Senapati *et al.,* [42] provided a list of additional AOs based on IVIFSs and expanded the theory of AA-TNM and AA-TCNM. In the scope of PFSs, Naeem *et al.,* [43] expanded the ideas of AA-TNM and AA-TCNM with more detailed information. Hussain *et al.,* [44] expanded on the ideas behind T-SFSs and created a few new AOs by utilizing the fundamental AA-TNM and AA-TCNM procedures. Recent advancements in MCDM highlight the integration of fuzzy models to handle uncertainty effectively. For example, Basuri *et al.,* [45] proposed an Entropy–VIKOR approach using generalized pentagonal intuitionistic fuzzy numbers for ranking higher education institutions. Their work motivates the development of more robust and flexible decision-making frameworks in this study. A MCDM approach developed by Duymaz *et al.,* [46], provides a systematic framework for evaluating the financial performance of energy firms listed on Borsa Istanbul (BIST) by considering multiple financial indicators simultaneously. It enables decision-makers to rank companies objectively by integrating criteria such as profitability, liquidity, and efficiency under uncertainty. Such approaches enhance the reliability and comprehensiveness of financial analysis in the energy sector. Karamat and Sarfraz [47] applied a MADM framework using complex Pythagorean fuzzy data with prioritized Aczel–Alsina aggregation operators to evaluate a software company, demonstrating how such approaches can effectively capture both the uncertainty of expert judgments and the relative importance of criteria. In the context of evaluating complex service systems, advanced fuzzy MCDM methods have proven highly effective in handling uncertainty and subjective judgments. For instance, Li *et al.,* [48] proposed an enhanced spherical cubic fuzzy WASPAS method to assess the service quality of a crowdsourcing logistics platform, demonstrating the ability of such models to capture both quantitative and qualitative criteria simultaneously. Gupta *et al.,* [49] introduces efficient techniques for handling nonlinear problems under bipolar fuzzy environments. It employs the Fuzzy Adomian Decomposition Method for analytical approximation and the Fuzzy Newton–Raphson Method for fast numerical convergence. Gazi *et al.,* [50] applies multi-criteria decision-making techniques under uncertainty to analyze psychological health across age groups. It incorporates fuzzy or uncertain data to evaluate and rank age groups based on the severity of various psychological issues. Mandal *et al.,* [51] proposes an advanced fuzzy MCDM framework to handle uncertainty in mental health assessment. It utilizes interval-valued Pythagorean trapezoidal fuzzy numbers to model imprecise clinical information and improve decision accuracy. The approach effectively supports the diagnosis and prioritization of psychiatric disorders, offering a reliable tool for healthcare decision-making. Gazi *et al.,* [52] presents an efficient decision-making framework for optimal site selection under uncertainty. It integrates MCDM methods with a novel de-i-fuzzification technique to convert fuzzy information into precise values for better evaluation. Adak *et al.,* [53] introduces a novel approach for handling assignment problems under uncertainty. It employs Pythagorean fuzzy sets with spherical distance measures to effectively evaluate and allocate resources. The framework enhances decision accuracy and provides a robust tool for solving complex management optimization problems. Recent advancements in fuzzy decision-making have emphasized the need for more expressive frameworks capable of handling complex and uncertain information. In this context, Ullah *et al.,* [54] proposed circular intuitionistic fuzzy power Muirhead-mean-based Aczel–Alsina aggregation operators within the framework of circular intuitionistic fuzzy sets. The introduced model extends traditional intuitionistic fuzzy sets by incorporating a circular structure characterized by radius and hesitation degrees, thereby enhancing the representation of uncertainty.

1.1 Research Gap

Despite significant advancements in fuzzy set theory, existing models such as FSs, IFSs, PyFSs, and q-ROFSs rely on real-valued membership and non-membership grades, limiting their ability to capture multidimensional and phase-based uncertainty. In many real-world problems, uncertainty involves both magnitude and phase, which these models cannot fully represent. Although CFSs address phase information, they still lack flexibility in handling higher uncertainty and Pythagorean constraints. In particular, they fail to model phase relationships between membership and non-membership grades. Therefore, a gap exists in integrating complex-valued representation with Pythagorean structure. CPyFSs can address this by offering a more flexible and comprehensive framework for handling dynamic and phase-dependent decision environments. Aczel-Alsina aggregation operators are unable to reduce the negative characteristics of complex data or examine the links between input arguments. Heronian Means operators are capable of forming interrelationship. The literature shows that aggregation operators combining Aczel–Alsina (AA) operational laws with HM and GHM operators have not yet been developed. Such operators can reduce the impact of irregular data while considering the relationships among input arguments.

1.2 Research Questions

Based on the identified gap, this study addresses the following questions:

i. In what way can Aczel-Alsina Heronian mean and Aczel-Alsina geometric Heronian mean aggregation operators be generalized under the CPyFS framework?
ii. How can the proposed operators be verified to satisfy essential mathematical properties such as idempotency, boundedness, and monotonicity?
iii. How can the MCDM methods be adapted within the CPyF environment to enhance decision-making effectiveness?
iv. How can the stability and reliability of the hybrid model be evaluated under varying parameters using sensitivity and comparative analysis?

1.3 Motivation and Objectives of Manuscript

Decision-making in real-world environments often involves highly uncertain, multidimensional, and phase-dependent information, which cannot be effectively handled by traditional fuzzy models. Existing frameworks such as FS, IFS, PyFS, and q-ROFS are limited because they rely only on real-valued membership and non-membership degrees, making them insufficient for modeling periodicity and phase information. Although CFSs incorporate phase information, they lack the ability to handle Pythagorean constraints and higher uncertainty flexibility. The main objectives of this study are as follows:

i. To construct the CPyF Aczel–Alsina Heronian mean (CPyFAAHM), CPyF Aczel–Alsina geometric Heronian mean (CPyFAAGHM), CPyF Aczel–Alsina weighted Heronian mean (CPyFAAWHM), and CPyF Aczel–Alsina weighted geometric Heronian mean (CPyFAAWGHM) operators.
ii. To prove and confirm the basic mathematical properties of the proposed aggregation operators, such as idempotency and boundedness, as well as monotonicity, and to assert their theoretical soundness.
iii. To develop MCDM models based on the weighted forms of these aggregation operators for handling CPyF information.
iv. To assess the effectiveness and practicality of the proposed model, a numerical example is presented for evaluating sustainable alternatives.
v. To assess the stability and reliability of the proposed model through sensitivity analysis and comparative studies.

2. Preliminaries

In this section, we discuss some basic definitions and fundamental properties.

Definition 1 [55]. A fuzzy set $\underbrace{ß}$ on a UOD X is defined as:

$$\underbrace{ß} = \{(x, \mu_{\underbrace{ß}}(x)) : x \in X\} \quad (1)$$

Where $\mu_{\underbrace{ß}}(x) \in [0, 1]$ represent a membership degree such that $0 \le \mu_{\underbrace{ß}}(x) \le 1$.

Definition 2 [56]. A Pythagorean fuzzy sets $\underbrace{ß}$ on a UOD X is defined as:

$$\underbrace{æ} = \{(\mu_{\underbrace{æ}}(x), ¥_{\underbrace{æ}}(x)) : x \in X\} \quad (2)$$

Where $\mu_{\underbrace{æ}}(x), ¥_{\underbrace{æ}}(x) \in [0, 1]$ represent a membership and non-membership degree such that $0 \le \mu_{\underbrace{æ}}(x), ¥_{\underbrace{æ}}(x) \le 1$.

Definition 3 [57]. Let a complex fuzzy set (CFS) $\underbrace{ß}$ be defined on X as:

$$\underbrace{ß} = \{(x, \mu_{\underbrace{ß}}(x)e^{2\pi i\left(\alpha_{\underbrace{ß}}(x)\right)}) : x \in X\},\ i = \sqrt{-1} \quad (3)$$

$\mu_{\underbrace{ß}}(x) \in [0, 1]$ and $\alpha_{\underbrace{ß}}(x) \in [0, 2\pi]$ serve as a MV of amplitude term along with phase term of $\underbrace{ß}$. A CFS has to meet the prerequisite:

$$0 \le \mu_{\underbrace{ß}}(x) \le 1 \text{ as well as } 0 \le \alpha_{\underbrace{ß}}(x) \le 1.$$

Definition 4 [58]. A CPyFS $\underbrace{æ}$ is expressed as on X:

$$\underbrace{æ} = \{\mu_{\underbrace{æ}}(x)e^{2\pi i\left(\alpha_{\underbrace{æ}}(x)\right)}, ¥_{\underbrace{æ}}(x)e^{2\pi i\left(\beta_{\underbrace{æ}}(x)\right)}) : x \in X\},\ i = \sqrt{-1} \quad (4)$$

$\mu_{\underbrace{æ}}(x) \in [0, 1]$ and $\alpha_{\underbrace{æ}}(x) \in [0, 2\pi]$ serve as MV of $\underbrace{æ}$. Similarly, $¥_{\underbrace{æ}}(x) \in [0, 1]$ and $\beta_{\underbrace{æ}}(x) \in [0, 2\pi]$ serve as NMV of $\underbrace{æ}$. A CPyFS has to meet the prerequisite:

$$0 \le \mu^2_{\underbrace{æ}}(x) + ¥^2_{\underbrace{æ}}(x) \le 1 \text{ and } 0 \le \alpha^2_{\underbrace{æ}}(x) + \beta^2_{\underbrace{æ}}(x) \le 1.$$

Definition 5 [59]. Here is how Aczel Alsina T-N is conceptualized:

$$\Gamma_{\varsigma}^{y}(æ_1, æ_2) = \begin{cases} \Gamma_{V}(æ_1, æ_2) \text{ if } y = 0 \\ \min(æ_1, æ_2) \text{ if } y = \infty & \forall, 0 \le y \le +\infty \\ \exp^{-((-\log æ_1)^{y} + (-\log æ_2)^{y})^{\frac{1}{y}}} & \text{otherwise} \end{cases} \quad (5)$$

In addition, the Aczel Alsina T-CN is conceptualized as:

$$\delta_{\varsigma}^{y}(æ_1, æ_2) = \begin{cases} \delta_{V}(æ_1, æ_2) \text{ if } y = 0 \\ \max(æ_1, æ_2) \text{ if } y = \infty & \forall, 0 \le y \le +\infty \\ 1 - \exp^{-((-\log æ_1)^{y} + (-\log æ_2)^{y})^{\frac{1}{y}}} & \text{otherwise} \end{cases} \quad (6)$$

Definition 6 [60]. Let $æ = (\mu_{æ}(x)e^{2\pi i(\alpha_{æ}(x))}, ¥_{æ}(x)e^{2\pi i(\beta_{æ}(x))})$ be a CPyFV, then the score $\bar{\bar{C}}(æ)$ and accuracy $\bar{\bar{ƙ}}(æ)$ functions are expressed as:

$$\bar{\bar{C}}(æ) = \frac{(\mu(x))^2 - (¥(x))^2 + (\alpha(x))^2 - (\beta(x))^2}{2} \text{ and } \bar{\bar{ƙ}}(æ) = \frac{(\mu(x))^2 + (¥(x))^2 + (\alpha(x))^2 + (\beta(x))^2}{2},$$

Where, $\bar{\bar{C}}(æ) \in [-1, 1]$ and $\bar{\bar{ƙ}}(æ) \in [0, 1]$.

Definition 7 [61]. Let $æ_1 = (\mu_{æ_1}(x)e^{2\pi i(\alpha_{æ_1}(x))}, ¥_{æ_1}(x)e^{2\pi i(\beta_{æ_1}(x))})$ and $æ_2 = (\mu_{æ_2}(x)e^{2\pi i(\alpha_{æ_2}(x))}, ¥_{æ_2}(x)e^{2\pi i(\beta_{æ_2}(x))})$ are two CPyFVs, then

i. If $\bar{\bar{C}}(æ_1) < \bar{\bar{C}}(æ_2)$, then $æ_1 < æ_2$
ii. If $\bar{\bar{C}}(æ_1) = \bar{\bar{C}}(æ_2)$, at that point find accuracy function
   a) If $\bar{\bar{ƙ}}(æ_1) < \bar{\bar{ƙ}}(æ_2)$, then $æ_1 < æ_2$
   b) If $\bar{\bar{ƙ}}(æ_1) = \bar{\bar{ƙ}}(æ_2)$, then $æ_1 = æ_2$.

Definition 8 [62]. Let $æ_1 = (\mu_{æ_1}(x)e^{2\pi i(\alpha_{æ_1}(x))}, ¥_{æ_1}(x)e^{2\pi i(\beta_{æ_1}(x))})$, $æ_2 = (\mu_{æ_2}(x)e^{2\pi i(\alpha_{æ_2}(x))}, ¥_{æ_2}(x)e^{2\pi i(\beta_{æ_2}(x))})$, are two CPyFVs, then the following operation holds:

i. $æ_1 \sqsubseteq æ_2$ if $\mu_1 \le \mu_2$, $\alpha_1 \le \alpha_2$, $¥_1 \ge ¥_2$ and $\beta_1 \ge \beta_2$
ii. $æ_1 \sqsubseteq æ_2$ if $\mu_1 = \mu_2$, $\alpha_1 = \alpha_2$, $¥_1 = ¥_2$ and $\beta_1 = \beta_2$
iii. $æ_1^{c} = (¥_{æ_1}(x)e^{2\pi i(\beta_{æ_1}(x))}, \mu_{æ_1}(x)e^{2\pi i(\alpha_{æ_1}(x))})$.

We review the basic axioms of the HM operator theory. Additionally, we examine GHM operators and its fundamental axioms.

Definition 9 [62]. Let $æ_i = (\mu_{æ_i}(x)e^{2\pi i(\alpha_{æ_i}(x))}, ¥_{æ_i}(x)e^{2\pi i(\beta_{æ_i}(x))})$, as a family of real numbers with $ƙ>0$ and $y>0$. Then, HMO is expressed as follows:

$$\mathrm{HM}(æ_1, æ_2, \dots, æ_n) = \left(\left(\frac{2}{n(n+1)}\right) \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{(æ_i^{ƙ} \times æ_j^{y})} \right)^{1/ƙ+y} \quad (7)$$

HM has to meet the prerequisite:

i. If all $æ_i = 0$, $\forall$ i, then the $\mathrm{HM}^{ƙ,y}(0,0,\dots,0) = 0$
ii. If all $æ_i = æ$, $\forall$ i, then the $\mathrm{HM}^{ƙ,y}(æ_1, æ_2, \dots, æ_n) = æ$

iii. If all $\underbrace{æ}_i \geq \bar{\bar{œ}}_i$, $\forall$ i, then the $HM^{\underbrace{k},\underbrace{y}}(\underbrace{æ}_1, \underbrace{æ}_2, \ldots, \underbrace{æ}_n) \geq HM^{\underbrace{k},\underbrace{y}}(\bar{\bar{œ}}_1, \bar{\bar{œ}}_2, \ldots, \bar{\bar{œ}}_n)$

iv. $\min\{\underbrace{æ}_i\} \leq HM^{\underbrace{k},\underbrace{y}}(\underbrace{æ}_1, \underbrace{æ}_2, \ldots, \underbrace{æ}_n) \leq \max\{\underbrace{æ}_i\}$.

Definition 10 [62]. Let $\underbrace{æ}_i = (\mu_{\underbrace{æ}_i}(x)e^{2\pi i\left(\alpha_{\underbrace{æ}_i}(x)\right)}, ¥_{\underbrace{æ}_i}(x)e^{2\pi i\left(\beta_{\underbrace{æ}_i}(x)\right)})$, is a family of real numbers with $\underbrace{k}>0$ and $\underbrace{y}>0$, then the GHM operator is given as:

$$GHM^{\underbrace{k},\underbrace{y}}\left(\underbrace{æ}_1, \underbrace{æ}_2, \ldots, \underbrace{æ}_n\right) = \left(\left(\frac{1}{\underbrace{k}+\underbrace{y}}\right)\prod_{i=1}^{n}\prod_{j=i}^{n}\left(\underbrace{k}\underbrace{æ}_i + \underbrace{y}\underbrace{æ}_i\right)\right)^{\frac{2}{n(n+1)}} \tag{8}$$

GHM has to meet the prerequisite:

i. If all $\underbrace{æ}_i = 0$, $\forall$ i, then the $GHM^{\underbrace{k},\underbrace{y}}(0,0,\ldots,0) = 0$

ii. If all $\underbrace{æ}_i = æ$, $\forall$ i, then the $GHM^{\underbrace{k},\underbrace{y}}(\underbrace{æ}_1, \underbrace{æ}_2, \ldots, \underbrace{æ}_n) = æ$

iii. If all $\underbrace{æ}_i \geq \underbrace{œ}_i$, $\forall$ i, then the $GHM^{k,y}(\underbrace{æ}_1, \underbrace{æ}_2, \ldots, \underbrace{æ}_n) \geq GHM^{k,y}(\underbrace{œ}_1, \underbrace{œ}_2, \ldots, \underbrace{œ}_n)$

iv. $\min\{\underbrace{æ}_i\} \leq GHM^{k,y}(\underbrace{æ}_1, \underbrace{æ}_2, \ldots, \underbrace{æ}_n) \leq \max\{\underbrace{æ}_i\}$.

3. Aczel Alsina Operations

The CPyF information system is examined in relation to AA operations, specifically the algebraic sum, product, scalar multiplication, along with power rule.

Definition 11 [62]. Let $\underbrace{æ} = (\mu_{\underbrace{æ}}(x)e^{2\pi\iota\left(\alpha_{\underbrace{æ}}(x)\right)}, ¥_{\underbrace{æ}}(x)e^{2\pi\iota\left(\beta_{\underbrace{æ}}(x)\right)})$, $\underbrace{æ}_1 = (\mu_{\underbrace{æ}_1}(x)e^{2\pi\iota\left(\alpha_{\underbrace{æ}_1}(x)\right)}, ¥_{\underbrace{æ}_1}(x)e^{2\pi\iota\left(\beta_{\underbrace{æ}_1}(x)\right)})$, $\underbrace{æ}_2 = (\mu_{\underbrace{æ}_2}(x)e^{2\pi\iota\left(\alpha_{\underbrace{æ}_2}(x)\right)}, ¥_{\underbrace{æ}_2}(x)e^{2\pi\iota\left(\beta_{\underbrace{æ}_2}(x)\right)})$, three CPyFVs, Ӯ ≥ 1 and λ > 0, then

i. $\underbrace{æ}_1 \oplus \underbrace{æ}_2 =$

$$\left\langle \sqrt{1 - e^{-\left(\left(-\ln\left(1-\mu^2_{\underbrace{æ}_1}\right)\right)^{y} + \left(-\ln\left(1-\mu^2_{\underbrace{æ}_2}\right)\right)^{y}\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{1-e^{-\left(\left(-\ln\left(1-\alpha^2_{\underbrace{æ}_1}\right)\right)^{y} + \left(-\ln\left(1-\alpha^2_{\underbrace{æ}_2}\right)\right)^{y}\right)^{\frac{1}{y}}}}\right)}, \right.$$

$$\left. \sqrt{e^{-\left(\left(-\ln\left(¥^2_{\underbrace{æ}_1}\right)\right)^{y} + \left(-\ln\left(¥^2_{\underbrace{æ}_2}\right)\right)^{y}\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{e^{-\left(\left(-\ln\left(\beta^2_{\underbrace{æ}_1}\right)\right)^{y} + \left(-\ln\left(\beta^2_{\underbrace{æ}_2}\right)\right)^{y}\right)^{\frac{1}{y}}}}\right)} \right\rangle.$$

ii. $$\underbrace{æ}_1 \otimes \underbrace{æ}_2 = \left\langle \sqrt{e^{-\left(\left(-\ln\left(\mu^2_{\underbrace{æ}_1}\right)\right)^{y} + \left(-\ln\left(\mu^2_{\underbrace{æ}_2}\right)\right)^{y}\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{e^{-\left(\left(-\ln\left(\alpha^2_{\underbrace{æ}_1}\right)\right)^{y} + \left(-\ln\left(\alpha^2_{\underbrace{æ}_2}\right)\right)^{y}\right)^{\frac{1}{y}}}}\right)}, \right.$$

$$\sqrt{1-e^{-\left(\left(-\ln\left(1-¥^2_{æ_1}\right)\right)^{y}+\left(-\ln\left(1-¥^2_{æ_1}\right)\right)^{y}\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{1-e^{-\left(\left(-\ln\left(1-\beta^2_{æ_1}\right)\right)^{y}+\left(-\ln\left(1-\beta^2_{æ_2}\right)\right)^{y}\right)^{\frac{1}{y}}}}\right)}\Bigg\rangle.$$

iii. $\lambda æ =$

$$\left\langle\sqrt{1-e^{-\left(\lambda\left(-\ln\left(1-\mu^2_{æ}\right)\right)^{y}\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{1-e^{-\left(\lambda\left(-\ln\left(1-\alpha^2_{æ}\right)\right)^{y}\right)^{\frac{1}{y}}}}\right)}, \sqrt{e^{\left(\lambda\left(-\ln\left(¥^2_{æ}\right)\right)^{y}\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{e^{\left(\lambda\left(-\ln\left(\beta^2_{æ}\right)\right)^{y}\right)^{\frac{1}{y}}}}\right)}\right\rangle$$

iv. $æ\lambda =$

$$\left\langle\sqrt{e^{-\left(\lambda\left(-\ln\left(\mu^2_{æ}\right)\right)^{y}\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{e^{\left(\lambda\left(-\ln\left(\alpha^2_{æ}\right)\right)^{y}\right)^{\frac{1}{y}}}}\right)}, \sqrt{1-e^{-\left(\lambda\left(-\ln\left(1-¥^2_{æ}\right)\right)^{y}\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{1-e^{-\left(\lambda\left(-\ln\left(1-\beta^2_{æ}\right)\right)^{y}\right)^{\frac{1}{y}}}}\right)}\right\rangle.$$

4. Complex Pythagorean Fuzzy AAWHM Operators

Using Aczel Alsina operational laws, we established a few new operators in this section. We also outline some aspects of our suggested methods for determining AOs flexibility.

Definition 12 [59]. If we define $æ_i = (\mu_{æ_i}(x)e^{2\pi\iota\left(\alpha_{æ_i}(x)\right)}, ¥_{æ_i}(x)e^{2\pi\iota\left(\beta_{æ_i}(x)\right)})$, is a family of real numbers with $k>0$ and $y>0$, then definition of the CPyFAAHMO is as follows:

$$\mathrm{CPyFAAHM}^{k,y}(æ_1, æ_2, \ldots, æ_n) = \left(\left(\frac{2}{n(n+1)}\right)\sum_{i=1}^{n}\sum_{j=1}^{n}\left(æ_i^{k}\times æ_j^{y}\right)\right)^{1/k+y} \tag{9}$$

Theorem 1. Let $æ_i = (\mu_{æ_i}(x)e^{2\pi\iota\left(\alpha_{æ_i}(x)\right)}, ¥_{æ_i}(x)e^{2\pi\iota\left(\beta_{æ_i}(x)\right)})$, as a family of CPyFVs with $k>0$ and $y>0$ then CPyFAAHMO is given as:

$$\mathrm{CPyFAAHM}^{k,y}\left(æ_1, æ_2, \ldots, æ_n\right) = \left\langle\sqrt{e^{-\left(\left(\frac{1}{k+y}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(k\left(-\ln\left(\mu^2_{æ_i}\right)\right)^{y}+y\left(-\ln\left(\mu^2_{æ_j}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\right.$$

$$e^{2\pi\iota\left(\sqrt{e^{-\left(\left(\frac{1}{k+y}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(k\left(-\ln\left(\alpha^2_{æ_i}\right)\right)^{y}+y\left(-\ln\left(\alpha^2_{æ_j}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\right)},$$

$$\sqrt{1-e^{-\left(\left(\frac{1}{\text{ķ}+\text{ɏ}}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(\text{ķ}\left(-\ln\left(1-\text{¥}_{\text{æ}_i}^2\right)\right)^{y}+\text{ɏ}\left(-\ln\left(1-\text{¥}_{\text{æ}_j}^2\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{1-e^{-\left(\left(\frac{1}{\text{ķ}+\text{ɏ}}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(\text{ķ}\left(-\ln\left(1-\beta_{\text{æ}_i}^2\right)\right)^{y}+\text{ɏ}\left(-\ln\left(1-\beta_{\text{æ}_j}^2\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\right)}\Bigg\rangle.$$

Proof.

Consider $\text{æ}_i=\left(\mu_{\text{æ}_i}(x)e^{2\pi\iota\left(\alpha_{\text{æ}_i}(x)\right)}, \text{¥}_{\text{æ}_i}(x)e^{2\pi\iota\left(\beta_{\text{æ}_i}(x)\right)}\right)$, as a family of CPyFVs with $\text{ķ}>0$, $\text{ɏ}>0$, we have,

$$\text{æ}_i^{\text{ķ}}=$$

$$\left\langle\sqrt{e^{-\left(\text{ķ}\left(-\ln\left(\mu_{\text{æ}_i}^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{e^{-\left(\text{ķ}\left(-\ln\left(\alpha_{\text{æ}_i}^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\right)} \sqrt{1-e^{-\left(\text{ķ}\left(-\ln\left(1-\text{¥}_{\text{æ}_i}^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{1-e^{-\left(\text{ķ}\left(-\ln\left(1-\beta_{\text{æ}_i}^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\right)}\right\rangle$$

$$\text{æ}_j^{\text{ɏ}}$$

$$=\left\langle\sqrt{e^{-\left(\text{ɏ}\left(-\ln\left(\mu_{\text{æ}_j}^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{e^{-\left(\text{ɏ}\left(-\ln\left(\alpha_{\text{æ}_j}^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\right)}, \sqrt{1-e^{-\left(\text{ɏ}\left(-\ln\left(1-\text{¥}_{\text{æ}_j}^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{1-e^{-\left(\text{ɏ}\left(-\ln\left(1-\beta_{\text{æ}_j}^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\right)}\right\rangle$$

$$\text{æ}_i^{\text{ķ}}\otimes\text{æ}_j^{\text{ɏ}}=\left\langle\sqrt{e^{-\left(\text{ķ}\left(-\ln\left(\mu_{\text{æ}_i}^2\right)\right)^{y}+\text{ɏ}\left(-\ln\left(\mu_{\text{æ}_j}^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{e^{-\left(\text{ķ}\left(-\ln\left(\alpha_{\text{æ}_i}^2\right)\right)^{y}+\text{ɏ}\left(-\ln\left(\alpha_{\text{æ}_j}^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\right)},\right.$$

$$\left.\sqrt{1-e^{-\left(\text{ķ}\left(-\ln\left(1-\text{¥}_{\text{æ}_i}^2\right)\right)^{y}+\text{ɏ}\left(-\ln\left(1-\text{¥}_{\text{æ}_j}^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{1-e^{-\left(\text{ķ}\left(-\ln\left(1-\beta_{\text{æ}_i}^2\right)\right)^{y}+\text{ɏ}\left(-\ln\left(1-\beta_{\text{æ}_j}^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\right)}\right\rangle$$

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Let $z = 1 - e^{-\left(\underbrace{k}\left(-\ln\left(1-¥^2_{\underbrace{æ}_i}\right)\right)^y + \underbrace{y}\left(-\ln\left(1-¥^2_{\underbrace{æ}_j}\right)\right)^y\right)^{\frac{1}{y}}}$ and ln(1-z)= $-\left(\underbrace{k}\left(-\ln\left(1-¥^2_{\underbrace{æ}_i}\right)\right)^y + \underbrace{y}\left(-\ln\left(1-¥^2_{\underbrace{æ}_j}\right)\right)^y\right)^{\frac{1}{y}}$, using this we have,

$$\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{æ}_i^{\underbrace{k}} \otimes \underbrace{æ}_j^{\underbrace{y}}\right)$$

$$=\left\langle \sqrt{1-e^{-\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(\mu^2_{\underbrace{æ}_i}\right)\right)^y + \underbrace{y}\left(-\ln\left(\mu^2_{\underbrace{æ}_j}\right)\right)^y\right)\right)^{\frac{1}{y}}}}\; e^{2\pi\iota\left(\sqrt{1-e^{-\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(\alpha^2_{\underbrace{æ}_i}\right)\right)^y + \underbrace{y}\left(-\ln\left(\alpha^2_{\underbrace{æ}_j}\right)\right)^y\right)\right)^{\frac{1}{y}}}}\right)},\right.$$

$$\left.\sqrt{e^{-\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-¥^2_{\underbrace{æ}_i}\right)\right)^y + \underbrace{y}\left(-\ln\left(1-¥^2_{\underbrace{æ}_j}\right)\right)^y\right)\right)^{\frac{1}{y}}}}\; e^{2\pi\iota\left(\sqrt{e^{-\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-\beta^2_{\underbrace{æ}_i}\right)\right)^y + \underbrace{y}\left(-\ln\left(1-\beta^2_{\underbrace{æ}_j}\right)\right)^y\right)\right)^{\frac{1}{y}}}}\right)}\right\rangle$$

$$\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\right)\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{æ}_i^{\underbrace{k}} \otimes \underbrace{æ}_j^{\underbrace{y}}\right) =$$

$$\left\langle \sqrt{1-e^{-\left(\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(\mu^2_{\underbrace{æ}_i}\right)\right)^y + \underbrace{y}\left(-\ln\left(\mu^2_{\underbrace{æ}_j}\right)\right)^y\right)\right)\right)^{\frac{1}{y}}}}\; e^{2\pi\iota\left(\sqrt{1-e^{-\left(\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(\alpha^2_{\underbrace{æ}_i}\right)\right)^y + \underbrace{y}\left(-\ln\left(\alpha^2_{\underbrace{æ}_j}\right)\right)^y\right)\right)\right)^{\frac{1}{y}}}}\right)},\right.$$

$$\left.\sqrt{e^{-\left(\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-¥^2_{\underbrace{æ}_i}\right)\right)^y + \underbrace{y}\left(-\ln\left(1-¥^2_{\underbrace{æ}_j}\right)\right)^y\right)\right)\right)^{\frac{1}{y}}}}\; e^{2\pi\iota\left(\sqrt{e^{-\left(\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-\beta^2_{\underbrace{æ}_i}\right)\right)^y + \underbrace{y}\left(-\ln\left(1-\beta^2_{\underbrace{æ}_j}\right)\right)^y\right)\right)\right)^{\frac{1}{y}}}}\right)}\right\rangle$$

So the final equation becomes

$$\left(\left(\frac{2}{n(n+1)}\right)\sum_{\substack{i=1\\ j=i}}^{n}\left(\ae_i^{\,k}\otimes \ae_j^{\,y}\right)\right)^{\frac{1}{k+y}} =$$

$$\left\langle \sqrt{e^{-\left(\left(\frac{1}{k+y}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(k\left(-\ln\left(\mu_{\ae_i}^2\right)\right)^{y}+y\left(-\ln\left(\mu_{\ae_j}^2\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\; e^{2\pi\iota\left(\sqrt{e^{-\left(\left(\frac{1}{k+y}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(k\left(-\ln\left(\alpha_{\ae_i}^2\right)\right)^{y}+y\left(-\ln\left(\alpha_{\ae_j}^2\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\right)},\right.$$

$$\left.\sqrt{1-e^{-\left(\left(\frac{1}{k+y}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(k\left(-\ln\left(1-¥_{\ae_i}^2\right)\right)^{y}+y\left(-\ln\left(1-¥_{\ae_j}^2\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\; e^{2\pi\iota\left(\sqrt{1-e^{-\left(\left(\frac{1}{k+y}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(k\left(-\ln\left(1-\beta_{\ae_i}^2\right)\right)^{y}+y\left(-\ln\left(1-\beta_{\ae_j}^2\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\right)}\right\rangle$$

Hence,
$CPyFAAHM^{k,y}(\ae_1,\ae_2,\ldots,\ae_n)=$

$$\left\langle \sqrt{e^{-\left(\left(\frac{1}{k+y}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(k\left(-\ln\left(\mu_{\ae_i}^2\right)\right)^{y}+y\left(-\ln\left(\mu_{\ae_j}^2\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\; e^{2\pi\iota\left(\sqrt{e^{-\left(\left(\frac{1}{k+y}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(k\left(-\ln\left(\alpha_{\ae_i}^2\right)\right)^{y}+y\left(-\ln\left(\alpha_{\ae_j}^2\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\right)},\right.$$

$$\left.\sqrt{1-e^{-\left(\left(\frac{1}{k+y}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(k\left(-\ln\left(1-¥_{\ae_i}^2\right)\right)^{y}+y\left(-\ln\left(1-¥_{\ae_j}^2\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\; e^{2\pi\iota\left(\sqrt{1-e^{-\left(\left(\frac{1}{k+y}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(k\left(-\ln\left(1-\beta_{\ae_i}^2\right)\right)^{y}+y\left(-\ln\left(1-\beta_{\ae_j}^2\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\right)}\right\rangle.$$

Theorem 2 (Idempotency). Let $\ae_i = \left(\mu_{\ae_i}(x)e^{2\pi\iota\left(\alpha_{\ae_i}(x)\right)}, ¥_{\ae_i}(x)e^{2\pi\iota\left(\beta_{\ae_i}(x)\right)}\right)$, $i = 1, 2, \ldots, n$ as assemblage of identical CPyFVs. Then $CPyFAAHM^{k,y}(\ae_1, \ae_2, \ldots, \ae_n) = \ae$.

Proof.

Since $\underbrace{æ}_i = (\mu_{\underbrace{æ}_i}(x)e^{2\pi\iota\left(\alpha_{\underbrace{æ}_i}(x)\right)}, ¥_{\underbrace{æ}_i}(x)e^{2\pi\iota\left(\beta_{\underbrace{æ}_i}(x)\right)}), \underbrace{æ} = (\mu_{\underbrace{æ}}(x)e^{2\pi\iota\left(\alpha_{\underbrace{æ}}(x)\right)}, ¥_{\underbrace{æ}}(x)e^{2\pi\iota\left(\beta_{\underbrace{æ}}(x)\right)})$, (i = 1, 2,…, n), and sprt ($\underbrace{æ}_i, \underbrace{æ}_j$) = 1 for all i, j = 1,2,…n then

$\text{CPyFAAHM}^{\underbrace{k},\underbrace{y}}(\underbrace{æ}_1, \underbrace{æ}_2, \ldots, \underbrace{æ}_n) = \text{CPyFAAHM}^{\underbrace{k},\underbrace{y}}(\underbrace{æ}, \underbrace{æ}, \ldots, \underbrace{æ})$

$$= \left\langle \sqrt{e^{-\left(\left(\frac{1}{\underbrace{k}+\underbrace{y}}\right)\left(\left(\frac{2}{\underbrace{n}(\underbrace{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}(-\ln(\mu^2))^y + \underbrace{y}(-\ln(\mu^2))^y\right)\right)\right)\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{e^{-\left(\left(\frac{1}{\underbrace{k}+\underbrace{y}}\right)\left(\left(\frac{2}{\underbrace{n}(\underbrace{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}(-\ln(\alpha^2))^y + \underbrace{y}(-\ln(\alpha^2))^y\right)\right)\right)\right)^{\frac{1}{y}}}}\right)}, \right.$$

$$\left. \sqrt{1 - e^{-\left(\left(\frac{1}{\underbrace{k}+\underbrace{y}}\right)\left(\left(\frac{2}{\underbrace{n}(\underbrace{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}(-\ln(1-¥^2))^y + \underbrace{y}(-\ln(1-¥^2))^y\right)\right)\right)\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{1-e^{-\left(\left(\frac{1}{\underbrace{k}+\underbrace{y}}\right)\left(\left(\frac{2}{\underbrace{n}(\underbrace{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}(-\ln(1-\beta^2))^y + \underbrace{y}(-\ln(1-\beta^2))^y\right)\right)\right)\right)^{\frac{1}{y}}}}\right)} \right\rangle$$

$$= \left\langle \sqrt{e^{-\left(\left(\frac{1}{\underbrace{k}+\underbrace{y}}\right)\left(\left(\frac{2}{\underbrace{n}(\underbrace{n}+1)}\right)\left(\underbrace{k}(-\ln(\mu^2))^y + \underbrace{y}(-\ln(\mu^2))^y\right)\right)\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{e^{-\left(\left(\frac{1}{\underbrace{k}+\underbrace{y}}\right)\left(\left(\frac{2}{\underbrace{n}(\underbrace{n}+1)}\right)\left(\underbrace{k}(-\ln(\alpha^2))^y + \underbrace{y}(-\ln(\alpha^2))^y\right)\right)\right)^{\frac{1}{y}}}}\right)}, \right.$$

$$\left. \sqrt{1 - e^{-\left(\left(\frac{1}{\underbrace{k}+\underbrace{y}}\right)\left(\left(\frac{2}{\underbrace{n}(\underbrace{n}+1)}\right)\left(\underbrace{k}(-\ln(1-¥^2))^y + \underbrace{y}(-\ln(1-¥^2))^y\right)\right)\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{1-e^{-\left(\left(\frac{1}{\underbrace{k}+\underbrace{y}}\right)\left(\left(\frac{2}{\underbrace{n}(\underbrace{n}+1)}\right)\left(\left(\underbrace{k}(-\ln(1-\beta^2))^y + \underbrace{y}(-\ln(1-\beta^2))^y\right)\right)\right)\right)^{\frac{1}{y}}}}\right)} \right\rangle$$

$$= \left\langle \sqrt{e^{-\left(\left(\frac{1}{\underbrace{k}+\underbrace{y}}\right)\left(\underbrace{k}(-\ln(\mu^2))^y + \underbrace{y}(-\ln(\mu^2))^y\right)\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{e^{-\left(\left(\frac{1}{\underbrace{k}+\underbrace{y}}\right)\left(\underbrace{k}(-\ln(\alpha^2))^y + \underbrace{y}(-\ln(\alpha^2))^y\right)\right)^{\frac{1}{y}}}}\right)}, \right.$$

$$\sqrt{1-e^{-\left(\left(\frac{1}{k+y}\right)\left(k(-\ln(1-\text{¥}^2))^y+y(-\ln(1-\text{¥}^2))^y\right)\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\left(\sqrt{1-e^{-\left(\left(\frac{1}{k+y}\right)\left(k\left(-\ln\left(1-\beta^2\right)\right)^y+y\left(-\ln\left(1-\beta^2\right)\right)^y\right)\right)^{\frac{1}{y}}}}\right)}\Big\rangle$$

$$=\Bigg\langle\sqrt{e^{-\left(\left(\frac{1}{k+y}\right)\left(k+y\right)(-\ln(\mu^2))^y\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\left(\sqrt{e^{-\left(\left(\frac{1}{k+y}\right)\left(k+y\right)\left(-\ln(\alpha^2)\right)^y\right)^{\frac{1}{y}}}}\right)},$$

$$\sqrt{1-e^{-\left(\left(\frac{1}{k+y}\right)\left(k+y\right)(-\ln(1-\text{¥}^2))^y\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\left(\sqrt{1-e^{-\left(\left(\frac{1}{k+y}\right)\left(k+y\right)\left(-\ln\left(1-\beta^2\right)\right)^y\right)^{\frac{1}{y}}}}\right)}\Bigg\rangle$$

$$=\Bigg\langle\sqrt{e^{-(-\ln(\mu^2))^y)^{\frac{1}{y}}}}\,e^{2\pi\iota\left(\sqrt{e^{-\left(\left(-\ln(\alpha^2)\right)^y\right)^{\frac{1}{y}}}}\right)},\sqrt{1-e^{-((-\ln(1-\text{¥}^2))^y)^{\frac{1}{y}}}}\,e^{2\pi\iota\left(\sqrt{1-e^{-\left(\left(-\ln\left(1-\beta^2\right)\right)^y\right)^{\frac{1}{y}}}}\right)}\Bigg\rangle$$

$$=\langle\sqrt{e^{\ln(\mu^2)}}\,e^{2\pi\iota\left(\sqrt{e^{\ln(\alpha^2)}}\right)},\sqrt{1-e^{\ln(1-\text{¥}^2)}}\,e^{2\pi\iota\left(\sqrt{1-e^{\ln\left(1-\beta^2\right)}}\right)}\rangle$$

$$=\left(\mu e^{2\pi\iota(\alpha)},\text{¥}e^{2\pi\iota(\beta)}\right).$$

Theorem 3 (Commutativity). Let $(\text{æ}_1{}', \text{æ}_2{}', \ldots, \text{æ}_n{}')$ be any permutation of $(\text{æ}_1, \text{æ}_2, \ldots, \text{æ}_n)$. Then,

$$\text{CPyFAAHM}^{k,y}(\text{æ}_1{}', \text{æ}_2{}', \ldots, \text{æ}_n{}') = \text{CPyFAAHM}^{k,y}(\text{æ}_1, \text{æ}_2, \ldots, \text{æ}_n).$$

Proof.

$(\text{æ}_1{}', \text{æ}_2{}', \ldots, \text{æ}_n{}')$ be any permutation of $(\text{æ}_1, \text{æ}_2, \ldots, \text{æ}_n)$ so,

$$\text{CPyFAAHM}^{k,y}(\text{æ}_1, \text{æ}_2, \ldots, \text{æ}_n) = \left(\left(\frac{2}{n(n+1)}\right)\sum_{\substack{i=1\\ j=i}}^{n}\left(\text{æ}_i{}^{k} \otimes \text{æ}_j{}^{y}\right)\right)^{\frac{1}{k+y}}$$

$$=\left(\left(\frac{2}{n(n+1)}\right)\sum_{\substack{i=1\\ j=i}}^{n}\left(\text{æ}_i{}'^{k} \otimes \text{æ}_j{}'^{k}\right)\right)^{\frac{1}{k+y}}$$

$$= \text{CPyFAAHM}^{k,y}(\text{æ}_1{}', \text{æ}_2{}', \ldots, \text{æ}_n{}').$$

Theorem 4 (Monotonicity). Let $\text{æ}_i = \left(\mu_{\text{æ}_i}(x)e^{2\pi\iota\left(\alpha_{\text{æ}_i}(x)\right)}, \text{¥}_{\text{æ}_i}(x)e^{2\pi\iota\left(\beta_{\text{æ}_i}(x)\right)}\right)$, and $\text{œ}_i = \left(\mu_{\text{œ}_i}(x)e^{2\pi\iota\left(\alpha_{\text{œ}_i}(x)\right)}, \text{¥}_{\text{œ}_i}(x)e^{2\pi\iota\left(\beta_{\text{œ}_i}(x)\right)}\right)$, $i = 1, 2, \ldots, n$ to be any two CPyFVs, if $\text{æ}_i \leq \text{œ}_i$, $\forall i$. Then $\text{CPyFAAHM}^{k,y}(\text{æ}_1, \text{æ}_2, \ldots, \text{æ}_n) \leq \text{CPyFAAHM}^{k,y}(\text{œ}_1, \text{œ}_2, \ldots, \text{œ}_n)$.

Proof.

Let $\underbrace{æ}_i = (\mu_{\underbrace{æ}_i}(x)e^{2\pi\iota(\alpha_{\underbrace{æ}_i}(x))}, ¥_{\underbrace{æ}_i}(x)e^{2\pi\iota(\beta_{\underbrace{æ}_i}(x))})$, and $\underbrace{œ}_i = (\mu_{\underbrace{œ}_i}(x)e^{2\pi\iota(\alpha_{\underbrace{œ}_i}(x))}, ¥_{\underbrace{œ}_i}(x)e^{2\pi\iota(\beta_{\underbrace{œ}_i}(x))})$, i = 1, 2, … ,n be the family of CPyFVs and $\underbrace{æ}_i \le \underbrace{œ}_i$, $\forall i$. Then

$\mathrm{CPyFAAHM}^{ķ,ỿ}(\underbrace{æ}_1, \underbrace{æ}_2, \dots, \underbrace{æ}_n) \le \mathrm{CPyFAAHM}^{ķ,ỿ}(\underbrace{œ}_1, \underbrace{œ}_2, \dots, \underbrace{œ}_n)$

=

$$\left\langle \sqrt{e^{-\left(\left(\frac{1}{ķ+ỿ}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(ķ\left(-\ln\left(\mu^2_{\underbrace{æ}_i}\right)\right)^{y}+ỿ\left(-\ln\left(\mu^2_{\underbrace{æ}_j}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{e^{-\left(\left(\frac{1}{ķ+ỿ}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(ķ\left(-\ln\left(\alpha^2_{\underbrace{æ}_i}\right)\right)^{y}+ỿ\left(-\ln\left(\alpha^2_{\underbrace{æ}_j}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\right)},\right.$$

$$\left.\sqrt{1-e^{-\left(\left(\frac{1}{ķ+ỿ}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(ķ\left(-\ln\left(1-¥^2_{\underbrace{æ}_i}\right)\right)^{y}+ỿ\left(-\ln\left(1-¥^2_{\underbrace{æ}_j}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{1-e^{-\left(\left(\frac{1}{ķ+ỿ}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(ķ\left(-\ln\left(1-\beta^2_{\underbrace{æ}}\right)\right)^{y}+ỿ\left(-\ln\left(1-\beta^2_{\underbrace{æ}_j}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\right)}\right\rangle$$

$\le$

$$\left\langle \sqrt{e^{-\left(\left(\frac{1}{ķ+ỿ}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(ķ\left(-\ln\left(\mu^2_{\underbrace{œ}_i}\right)\right)^{y}+ỿ\left(-\ln\left(\mu^2_{\underbrace{œ}_j}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{e^{-\left(\left(\frac{1}{ķ+ỿ}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(ķ\left(-\ln\left(\alpha^2_{\underbrace{œ}_i}\right)\right)^{y}+ỿ\left(-\ln\left(\alpha^2_{\underbrace{œ}_j}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\right)},\right.$$

$$\left.\sqrt{\mathbf{1-e}^{-\left(\left(\frac{1}{ķ+ỿ}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(ķ\left(-\ln\left(1-¥^2_{\underbrace{œ}_i}\right)\right)^{y}+ỿ\left(-\ln\left(1-¥^2_{\underbrace{œ}_j}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\left(\sqrt{1-e^{-\left(\left(\frac{1}{ķ+ỿ}\right)\left(\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(ķ\left(-\ln\left(1-\beta^2_{\underbrace{œ}_i}\right)\right)^{y}+ỿ\left(-\ln\left(1-\beta^2_{\underbrace{œ}_j}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\right)}\right\rangle$$

Hence,

$\mathrm{CPyFAAHM}^{ķ,ỿ}(\underbrace{æ}_1, \underbrace{æ}_2, \dots, \underbrace{æ}_n) \le \mathrm{CPyFAAHM}^{ķ,ỿ}(\underbrace{œ}_1, \underbrace{œ}_2, \dots, \underbrace{œ}_n)$.

Theorem 5 (Boundedness). Let $\underset{\smile}{\ae}_{\mathbf{i}} = \left(\mu_{\underset{\smile}{\ae}_i}(\mathbf{x})\mathbf{e}^{2\pi\iota\left(\alpha_{\underset{\smile}{\ae}_i}(\mathbf{x})\right)}, \text{¥}_{\underset{\smile}{\ae}_i}(\mathbf{x})\mathbf{e}^{2\pi\iota\left(\beta_{\underset{\smile}{\ae}_i}(\mathbf{x})\right)}\right)$, $\boldsymbol{i} = 1, 2, \ldots, n$ as a family of CPyFVs and $\underset{\smile}{\ae}_{\mathbf{i}}^{-} = \min\left(\underset{\smile}{\ae}_{\mathbf{1}}, \underset{\smile}{\ae}_{\mathbf{2}}, \ldots, \underset{\smile}{\ae}_{\mathbf{n}}\right)$ and $\underset{\smile}{\ae}_{\mathbf{i}}^{+} = \max\left(\underset{\smile}{\ae}_{\mathbf{1}}, \underset{\smile}{\ae}_{\mathbf{2}}, \ldots, \underset{\smile}{\ae}_{\mathbf{n}}\right)$ then,

$$\underset{\smile}{\ae}_i^{-} \leq \mathrm{CPyFAAHM}^{\underset{\smile}{k},\underset{\smile}{y}}\left(\underset{\smile}{\ae}_1, \underset{\smile}{\ae}_2, \ldots, \underset{\smile}{\ae}_n\right) \leq \underset{\smile}{\ae}_i^{+}.$$

Proof.

Suppose $\underset{\smile}{\ae}_i = \left(\mu_{\underset{\smile}{\ae}_i}(x)e^{2\pi\iota\left(\alpha_{\underset{\smile}{\ae}_i}(x)\right)}, \text{¥}_{\underset{\smile}{\ae}_i}(x)e^{2\pi\iota\left(\beta_{\underset{\smile}{\ae}_i}(x)\right)}\right)$, i = 1, 2, …, n is a family of CPyFVs and $\underset{\smile}{\ae}_i^{-} = \min\left(\underset{\smile}{\ae}_1, \underset{\smile}{\ae}_2, \ldots, \underset{\smile}{\ae}_n\right)$ and $\underset{\smile}{\ae}_i^{+} = \max\left(\underset{\smile}{\ae}_1, \underset{\smile}{\ae}_2, \ldots, \underset{\smile}{\ae}_n\right)$ Since,

$\mu^{+}_{\underset{\smile}{\ae}_i} = \max\left(\mu_{\underset{\smile}{\ae}_i}(x)e^{2\pi\iota\left(\alpha_{\underset{\smile}{\ae}_i}(x)\right)}\right)$, $\mu^{-}_{\underset{\smile}{\ae}_i} = \min\left(\mu_{\underset{\smile}{\ae}_i}(x)e^{2\pi\iota\left(\alpha_{\underset{\smile}{\ae}_i}(x)\right)}\right)$, $\text{¥}^{+}_{\underset{\smile}{\ae}_i} = \max\left(\text{¥}_{\underset{\smile}{\ae}_i}(x)e^{2\pi\iota\left(\beta_{\underset{\smile}{\ae}_i}(x)\right)}\right)$, and $\text{¥}^{-}_{\underset{\smile}{\ae}_i} = \min\left(\text{¥}_{\underset{\smile}{\ae}_i}(x)e^{2\pi\iota\left(\beta_{\underset{\smile}{\ae}_i}(x)\right)}\right)$

To show $\underset{\smile}{\ae}_i^{-} \leq \mathrm{CPyFAAHM}^{\underset{\smile}{k},\underset{\smile}{y}}\left(\underset{\smile}{\ae}_1, \underset{\smile}{\ae}_2, \ldots, \underset{\smile}{\ae}_n\right) \leq \underset{\smile}{\ae}_i^{+}$ we must satisfy the following inequalities

$$\left\langle \sqrt{e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(\mu^2_{\underset{\smile}{\ae}_i}-\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(\mu^2_{\underset{\smile}{\ae}_j}-\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\sqrt{e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(\alpha^2_{\underset{\smile}{\ae}_i}-\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(\alpha^2_{\underset{\smile}{\ae}_j}-\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}},\right.$$

$$\leq \sqrt{e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(\mu^2_{\underset{\smile}{\ae}_i}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(\mu^2_{\underset{\smile}{\ae}_j}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\, e^{2\pi\iota\sqrt{e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(\alpha^2_{\underset{\smile}{\ae}_i}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(\alpha^2_{\underset{\smile}{\ae}_j}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}}$$

$$\leq \sqrt{e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(\mu^2_{\underset{\smile}{\ae}_i}+\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(\mu^2_{\underset{\smile}{\ae}_i}+\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}$$

$$\left. e^{2\pi\iota\sqrt{e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(\alpha^2_{\underset{\smile}{\ae}_i}+\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(\alpha^2_{\underset{\smile}{\ae}_j}+\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}} \right\rangle$$

$$\left\langle \leq \sqrt{1-e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-\text{¥}^2_{\underset{\smile}{\ae}_i}-\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-\text{¥}^2_{\underset{\smile}{\ae}_j}-\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}},\right.$$

$$e^{2\pi\iota\sqrt{1-e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-\beta^2_{\underset{\smile}{\ae}_i}-\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-\beta^2_{\underset{\smile}{\ae}_j}-\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}}$$

$$\leq \sqrt{1-e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-¥^2_{æ_i}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-¥^2_{æ_j}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}$$
$$e^{2\pi\iota\sqrt{1-e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-\beta^2_{æ_i}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-\beta^2_{æ_j}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}}$$

$$\leq \sqrt{1-e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-¥^2_{æ_i+}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-¥^2_{æ_j+}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}$$
$$e^{2\pi\iota\sqrt{1-e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-\beta^2_{æ_i+}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-\beta^2_{æ_j+}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}}\Bigg\rangle$$

Hence

$$æ_i^{-} \leq CPyFAAHM^{\underset{\smile}{k},\underset{\smile}{y}}(æ_1, æ_2, \ldots, æ_n) \leq æ_i^{+}.$$

Definition 13 [62]. Let $æ_i = (\mu_{æ_i}(x)e^{2\pi\iota\left(\alpha_{æ_i}(x)\right)}, ¥_{æ_i}(x)e^{2\pi\iota\left(\beta_{æ_i}(x)\right)})$, be family of real numbers, with weight vector $\underset{\smile}{\vartheta} = (\underset{\smile}{\vartheta}_1, \underset{\smile}{\vartheta}_2, \ldots, \underset{\smile}{\vartheta}_n)^T$, $\vartheta_n \in [0, 1]$, $\sum_{i=1}^{n}\underset{\smile}{\vartheta}_i = 1$ and $\underset{\smile}{k} > 0$, $\underset{\smile}{y} > 0$. Then, a CPyFAAWHM is express as follows:

$$CPyFAAWHM^{\underset{\smile}{k},\underset{\smile}{y}}(æ_1, æ_2, \ldots, æ_n) = \left(\varsigma \sum_{\substack{i=1\\j=i}}^{n}(\underset{\smile}{\vartheta}_i\, æ_i)^{\underset{\smile}{k}} \otimes (\underset{\smile}{\vartheta}_j\, æ_j)^{\underset{\smile}{y}}\right)^{\wp} \quad (10)$$

Where $\wp = \frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}$ and $\varsigma = \frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}$.

Theorem 6. Let $æ_i = (\mu_{æ_i}(x)e^{2\pi\iota\left(\alpha_{æ_i}(x)\right)}, ¥_{æ_i}(x)e^{2\pi\iota\left(\beta_{æ_i}(x)\right)})$, be family of CPyFVs, $\underset{\smile}{\vartheta}_n \in [0, 1]$, $\sum_{i=1}^{n}\underset{\smile}{\vartheta}_i = 1$ and $\underset{\smile}{k} > 0$, $\underset{\smile}{y} > 0$. Then, a CPyFAAWHM is expressed as follows:

$CPyFAAWHM^{\underset{\smile}{k},\underset{\smile}{y}}(æ_1, æ_2, \ldots, æ_n) =$

$$\Bigg\langle\sqrt{e^{-\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\left(\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(1-\mu_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\sqrt{1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(1-\mu_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}$$
$$e^{2\pi\iota\sqrt{e^{-\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\left(\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(1-\alpha_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(1-\alpha_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}},$$

$$\sqrt{1-e^{-\left(\frac{1}{\underbrace{k}+\underbrace{y}}\left(\left(\frac{2}{\underbrace{n}(\underbrace{n}+1)}\right)\left(\Sigma_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln(¥_i^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_j\left(-\ln(¥_j^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}$$

$$e^{2\pi\iota\sqrt{1-e^{-\left(\frac{1}{\underbrace{k}+\underbrace{y}}\left(\left(\frac{2}{\underbrace{n}(\underbrace{n}+1)}\right)\left(\Sigma_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln(\beta_i^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_j\left(-\ln(\beta_j^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}}\Bigg\rangle.$$

Proof. Let $\underbrace{æ}_i = \left(\mu_{\underbrace{æ}_i}(x)e^{2\pi\iota\left(\alpha_{\underbrace{æ}_i}(x)\right)}, ¥_{\underbrace{æ}_i}(x)e^{2\pi\iota\left(\beta_{\underbrace{æ}_i}(x)\right)}\right)$, then

$$\underbrace{\vartheta}_i\,\underbrace{æ}_i = \left\langle\sqrt{1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln\left(1-\mu_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln\left(1-\alpha_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}}}, \sqrt{e^{-\left(\underbrace{\vartheta}_i\left(-\ln(¥_i^2)\right)^{y}\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{e^{-\left(\underbrace{\vartheta}_i\left(-\ln(\beta_i^2)\right)^{y}\right)^{\frac{1}{y}}}}}\right\rangle$$

$$\underbrace{\vartheta}_j\,\underbrace{æ}_j = \left\langle\sqrt{1-e^{-\left(\underbrace{\vartheta}_j\left(-\ln\left(1-\mu_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{1-e^{-\left(\underbrace{\vartheta}_j\left(-\ln\left(1-\alpha_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}}}, \sqrt{e^{-\left(\underbrace{\vartheta}_j\left(-\ln(¥_j^2)\right)^{y}\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{e^{-\left(\underbrace{\vartheta}_j\left(-\ln(\beta_i^2)\right)^{y}\right)^{\frac{1}{y}}}}}\right\rangle$$

$$\left(\underbrace{\vartheta}_i\,\underbrace{æ}_i\right)^{\underbrace{k}} = \left\langle\sqrt{e^{-\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln\left(1-\mu_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{e^{-\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln\left(1-\alpha_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)^{\frac{1}{y}}}}},\right.$$

$$\left.\sqrt{1-e^{-\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln(¥_i^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{1-e^{-\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln(\beta_i^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)^{\frac{1}{y}}}}}\right\rangle$$

$$\left(\underbrace{\vartheta}_j\,\underbrace{æ}_j\right)^{\underbrace{y}} = \left\langle\sqrt{e^{-\left(\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_j\left(-\ln\left(1-\mu_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{e^{-\left(\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_j\left(-\ln\left(1-\alpha_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)^{\frac{1}{y}}}}},\right.$$

$$\left.\sqrt{1-e^{-\left(\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_j\left(-\ln(¥_j^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{1-e^{-\left(\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_j\left(-\ln(\beta_j^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)^{\frac{1}{y}}}}}\right\rangle$$

$$\left(\underbrace{\vartheta}_{i}\underbrace{æ}_{i}\right)^{\underbrace{k}} \otimes \left(\underbrace{\vartheta}_{j}\underbrace{æ}_{j}\right)^{\underbrace{y}} =$$

$$\left\langle \sqrt{e^{-\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{i}\left(-\ln\left(1-\mu_i^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\vartheta_j\left(-\ln\left(1-\mu_j^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}}\, e^{2\pi\iota\sqrt{e^{-\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{i}\left(-\ln\left(1-\alpha_i^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{j}\left(-\ln\left(1-\alpha_j^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}}},\right.$$

$$\left.\sqrt{1-e^{-\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{i}\left(-\ln\left(¥_i^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{j}\left(-\ln\left(¥_j^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}}\, e^{2\pi\iota\sqrt{1-e^{-\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{i}\left(-\ln\left(\beta_i^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{j}\left(-\ln\left(\beta_j^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}}\right\rangle$$

Now let $z = 1 - e^{-\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{i}\left(-\ln\left(1-\mu_i^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{j}\left(-\ln\left(1-\mu_j^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}$

$$\ln(1-z) = -\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{i}\left(-\ln\left(1-\mu_i^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{j}\left(-\ln\left(1-\mu_j^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}$$

$$\sum_{\substack{i=1\\ j=i}}^{n}\left(\underbrace{\vartheta}_{i}\underbrace{æ}_{i}\right)^{\underbrace{k}} \otimes \left(\underbrace{\vartheta}_{j}\underbrace{æ}_{j}\right)^{\underbrace{y}} =$$

$$\left\langle \sqrt{1-e^{-\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{i}\left(-\ln\left(1-\mu_i^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{j}\left(-\ln\left(1-\mu_j^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}\right)\right)^{\frac{1}{\underbrace{y}}}}}\, e^{2\pi\iota\sqrt{1-e^{-\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{i}\left(-\ln\left(1-\alpha_i^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{j}\left(-\ln\left(1-\alpha_j^2\right)\right)^{y}\right)^{\frac{1}{\underbrace{y}}}}\right)\right)^{y}\right)\right)^{\frac{1}{\underbrace{y}}}}}},\right.$$

$$\sqrt{e^{-\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{i}\left(-\ln\left(¥_{i}^{2}\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}\left(-\ln\left(¥_{j}^{2}\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)^{\frac{1}{y}}}},\; e^{2\pi\iota\sqrt{e^{-\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{i}\left(-\ln\left(\beta_{i}^{2}\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{j}\left(-\ln\left(\beta_{j}^{2}\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)^{\frac{1}{y}}}}}\right\rangle$$

$$\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{\vartheta}_{i}\,\underbrace{æ}_{i}\right)^{\underbrace{k}}\otimes\left(\underbrace{\vartheta}_{j}\,\underbrace{æ}_{j}\right)^{\underbrace{y}}\right) =$$

$$\left\langle\sqrt{1-e^{-\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{i}\left(-\ln\left(1-\mu_{i}^{2}\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{j}\left(-\ln\left(1-\mu_{j}^{2}\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}},\; e^{2\pi\iota\sqrt{1-e^{-\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{i}\left(-\ln\left(1-\alpha_{i}^{2}\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{j}\left(-\ln\left(1-\alpha_{j}^{2}\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}}\right\rangle,$$

$$\left\langle\sqrt{e^{-\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{i}\left(-\ln\left(¥_{i}^{2}\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{j}\left(-\ln\left(¥_{j}^{2}\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}},\; e^{2\pi\iota\sqrt{e^{-\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{i}\left(-\ln\left(\beta_{i}^{2}\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{j}\left(-\ln\left(\beta_{j}^{2}\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}}\right\rangle$$

$$\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{\vartheta}_{i}\,\underbrace{æ}_{i}\right)^{\underbrace{k}}\otimes\left(\underbrace{\vartheta}_{j}\,\underbrace{æ}_{j}\right)^{\underbrace{y}}\right)\right)^{\frac{1}{\underbrace{k}+\underbrace{y}}} =$$

$$= \left\langle\sqrt{e^{-\left(\frac{1}{\underbrace{k}+\underbrace{y}}\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{i}\left(-\ln\left(1-\mu_{i}^{2}\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_{j}\left(-\ln\left(1-\mu_{j}^{2}\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\right.$$

$$e^{2\pi\iota\sqrt{e^{-\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(1-\alpha_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(1-\alpha_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}},$$

$$\sqrt{1-e^{-\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln(¥_i^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln(¥_j^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}$$

$$e^{2\pi\iota\sqrt{1-e^{-\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln(\beta_i^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln(\beta_j^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}}\Bigg\rangle$$

Hence

$CPyFAAWHM^{\underset{\smile}{k},\underset{\smile}{y}}(\underset{\smile}{æ}_1, \underset{\smile}{æ}_2, \ldots, \underset{\smile}{æ}_n) =$

$$\Bigg\langle\sqrt{e^{-\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(1-\mu_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(1-\mu_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}$$

$$e^{2\pi\iota\sqrt{e^{-\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(1-\alpha_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(1-\alpha_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}},$$

$$\sqrt{1-e^{-\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln(¥_i^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln(¥_j^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}$$

$$e^{2\pi\iota\sqrt{1-e^{-\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln(\beta_i^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln(\beta_j^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}}\Bigg\rangle.$$

Theorem 7 (Idempotency). Let $\underset{\smile}{æ}_i = (\mu_{\underset{\smile}{æ}_i}(x)e^{2\pi\iota\left(\alpha_{\underset{\smile}{æ}_i}(x)\right)}, ¥_{\underset{\smile}{æ}_i}(x)e^{2\pi\iota\left(\beta_{\underset{\smile}{æ}_i}(x)\right)})$, i = 1, 2, …, n as the collection of identical CPyFVs. Then, $CPyFAAWHM^{\underset{\smile}{k},\underset{\smile}{y}}(\underset{\smile}{æ}_1, \underset{\smile}{æ}_2, \ldots, \underset{\smile}{æ}_n) = \underset{\smile}{æ}$.

Proof.

The same applies to proof.

Theorem 8 (Commutativity). Let $(\text{æ}_1{}', \text{æ}_2{}', \ldots, \text{æ}_n{}')$ be any permutation of $(\text{æ}_1, \text{æ}_2, \ldots, \text{æ}_n)$. Then,

$$\text{CPyFAAWHM}^{k,y}(\text{æ}_1{}', \text{æ}_2{}', \ldots, \text{æ}_n{}') = \text{CPyFAAWHM}^{k,y}(\text{æ}_1, \text{æ}_2, \ldots, \text{æ}_n).$$

Theorem 9 (Monotonicity). Consider $\text{æ}_i = (\mu_{\text{æ}_i}(x)e^{2\pi\iota(\alpha_{\text{æ}_i}(x))}, ¥_{\text{æ}_i}(x)e^{2\pi\iota(\beta_{\text{æ}_i}(x))})$, and $\bar{\bar{\text{œ}}}_i = (\mu_{\bar{\bar{\text{œ}}}_i}(x)e^{2\pi\iota(\alpha_{\bar{\bar{\text{œ}}}_i}(x))}, ¥_{\bar{\bar{\text{œ}}}_i}(x)e^{2\pi\iota(\beta_{\text{œ}_i}(x))})$, i = 1, 2, …, n as any two CPyFVs , if $\text{æ}_i \leq \bar{\bar{\text{œ}}}_j$, $\forall i$. Then

$$\text{CPyFAAWHM}^{k,y}(\text{æ}_1, \text{æ}_2, \ldots, \text{æ}_n) \leq \text{CPyFAAWHM}^{k,y}(\text{œ}_1, \text{œ}_2, \ldots, \text{œ}_n).$$

Proof.

The same applies to proof.

Theorem 10 (Boundedness). Let $\text{æ}_i = (\mu_{\text{æ}_i}(x)e^{2\pi\iota(\alpha_{\text{æ}_i}(x))}, ¥_{\text{æ}_i}(x)e^{2\pi\iota(\beta_{\text{æ}_i}(x))})$, i = 1 ,2 ,…, n as the family of CPyFVs and $\text{æ}_i{}^- = \min(\text{æ}_1, \text{æ}_2, \ldots, \text{æ}_n)$ and $\text{æ}_i{}^+ = \max(\text{æ}_1, \text{æ}_2, \ldots, \text{æ}_n)$ then,

$$\text{æ}_i{}^- \leq \text{CPyFAAWHM}^{k,y}(\text{æ}_1, \text{æ}_2, \ldots, \text{æ}_n) \leq \text{æ}_i{}^+.$$

5. Complex Pythagorean Fuzzy AAGHM and WHM operators

By employing AA operations under the system IFSs, we have now expanded the theory of the GHM. Provide an example to evaluate the AOs competitiveness.

Definition 14 [62]. Let $\text{æ}_i = (\mu_{\text{æ}_i}(x)e^{2\pi\iota(\alpha_{\text{æ}_i}(x))}, ¥_{\text{æ}_i}(x)e^{2\pi\iota(\beta_{\text{æ}_i}(x))})$, as family of real numbers with k, $y > 0$ then CPyFAAGHM operator is defined as follows:

$$\text{CPyFAAGHM}^{k,y}(\text{æ}_1, \text{æ}_2, \ldots, \text{æ}_n) = \left(\frac{1}{k+y}\sum_{i=1}^{n}\sum_{j=1}^{n}(k\,\text{æ}_i \oplus y\,\text{æ}_j)^{\frac{2}{n(n+1)}}\right) \tag{11}$$

Theorem 11. Let $\text{æ}_i = (\mu_{\text{æ}_i}(x)e^{2\pi\iota(\alpha_{\text{æ}_i}(x))}, ¥_{\text{æ}_i}(x)e^{2\pi\iota(\beta_{\text{æ}_i}(x))})$, as a family of CPyFVs with k, $y > 0$ then CPyFAAGHMO is given as:

$\text{CPyFAAGHM}^{k,y}(\text{æ}_1, \text{æ}_2, \ldots, \text{æ}_n) =$

$$\left\langle \sqrt{1-e^{-\left(\frac{1}{k+y}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{n(n+1)}\left(k\left(-\ln(1-\mu_i^2)\right)^y + y\left(-\ln(1-\mu_j^2)\right)^y\right)\right)\right)\right)^{\frac{1}{y}}}}\; e^{2\pi\iota\sqrt{1-e^{-\left(\frac{1}{k+y}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{n(n+1)}\left(k\left(-\ln(1-\alpha_i^2)\right)^y + y\left(-\ln(1-\alpha_j^2)\right)^y\right)\right)\right)\right)^{\frac{1}{y}}}}}, \right.$$

$$\left. \sqrt{e^{-\left(\frac{1}{k+y}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{n(n+1)}\left(k\left(-\ln(¥_i^2)\right)^y + y\left(-\ln(¥_j^2)\right)^y\right)\right)\right)\right)^{\frac{1}{y}}}}\; e^{2\pi\iota\sqrt{e^{-\left(\frac{1}{k+y}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{n(n+1)}\left(k\left(-\ln(\beta_i^2)\right)^y + y\left(-\ln(\beta_i^2)\right)^y\right)\right)\right)\right)^{\frac{1}{y}}}}} \right\rangle.$$

*Proof***.**

Let $\text{æ}_i = (\mu_{\text{æ}_i}(x)e^{2\pi\iota(\alpha_{\text{æ}_i}(x))}, ¥_{\text{æ}_i}(x)e^{2\pi\iota(\beta_{\text{æ}_i}(x))})$, i = 1, 2 , … , n we have

$$\underbrace{\text{ķ}}\,\underbrace{\text{æ}}_{i}=\left\langle\sqrt{1-e^{-\left(\underbrace{\text{ķ}}\left(-\ln\left(1-\mu_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{1-e^{-\left(\underbrace{\text{ķ}}\left(-\ln\left(1-\alpha_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}}},\sqrt{e^{-\left(\underbrace{\text{ķ}}\left(-\ln\left(\text{¥}_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{e^{-\left(\underbrace{\text{ķ}}\left(-\ln\left(\beta_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}}}\right\rangle$$

$$\underbrace{y}\,\underbrace{\text{æ}}_{j}=\left\langle\sqrt{1-e^{-\left(\underbrace{y}\left(-\ln\left(1-\mu_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{1-e^{-\left(\underbrace{y}\left(-\ln\left(1-\alpha_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}}},\sqrt{e^{-\left(\underbrace{y}\left(-\ln\left(\text{¥}_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{e^{-\left(\underbrace{y}\left(-\ln\left(\beta_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}}}\right\rangle$$

$$\underbrace{\text{ķ}}\,\underbrace{\text{æ}}_{i}\oplus\underbrace{y}\,\underbrace{\text{æ}}_{j}=\left\langle\sqrt{1-e^{-\left(\underbrace{\text{ķ}}\left(-\ln\left(1-\mu_i^2\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-\mu_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{1-e^{-\left(\underbrace{\text{ķ}}\left(-\ln\left(1-\alpha_i^2\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-\alpha_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}}},\right.$$
$$\left.\sqrt{e^{-\left(\underbrace{\text{ķ}}\left(-\ln\left(\text{¥}_i^2\right)\right)^{y}+\underbrace{y}\left(-\ln\left(\text{¥}_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{e^{-\left(\underbrace{\text{ķ}}\left(-\ln\left(\beta_i^2\right)\right)^{y}+\underbrace{y}\left(-\ln\left(\beta_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}}}\right\rangle$$

$$\left(\underbrace{\text{ķ}}\,\underbrace{\text{æ}}_{i}\oplus\underbrace{y}\,\underbrace{\text{æ}}_{j}\right)^{\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}}=$$
$$\left\langle\sqrt{e^{-\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\underbrace{\text{ķ}}\left(-\ln\left(1-\mu_i^2\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-\mu_j^2\right)\right)^{y}\right)\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{e^{-\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\underbrace{\text{ķ}}\left(-\ln\left(1-\alpha_i^2\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-\alpha_j^2\right)\right)^{y}\right)\right)^{\frac{1}{y}}}}},\right.$$
$$\left.\sqrt{1-e^{-\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\underbrace{\text{ķ}}\left(-\ln\left(\text{¥}_i^2\right)\right)^{y}+\underbrace{y}\left(-\ln\left(\text{¥}_j^2\right)\right)^{y}\right)\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{1-e^{-\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\underbrace{\text{ķ}}\left(-\ln\left(\beta_i^2\right)\right)^{y}+\underbrace{y}\left(-\ln\left(\text{¥}_j^2\right)\right)^{y}\right)\right)^{\frac{1}{y}}}}}\right\rangle$$

$$\sum_{\substack{i=1\\j=i}}^{n}\left(\left(\underbrace{\text{ķ}}\,\underbrace{\text{æ}}_{i}\oplus\underbrace{y}\,\underbrace{\text{æ}}_{j}\right)^{\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}}\right)=$$
$$\left\langle\sqrt{e^{-\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\underbrace{\text{ķ}}\left(-\ln\left(1-\mu_i^2\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-\mu_j^2\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{e^{-\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\underbrace{\text{ķ}}\left(-\ln\left(1-\alpha_i^2\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-\alpha_j^2\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}},\right.$$
$$\left.\sqrt{1-e^{-\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\underbrace{\text{ķ}}\left(-\ln\left(\text{¥}_i^2\right)\right)^{y}+\underbrace{y}\left(-\ln\left(\text{¥}_j^2\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{1-e^{-\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\underbrace{\text{ķ}}\left(-\ln\left(\beta_i^2\right)\right)^{y}+\underbrace{y}\left(-\ln\left(\beta_j^2\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}}\right\rangle$$

$$\left(\frac{1}{\underbrace{\text{ķ}}+\underbrace{y}}\right)\sum_{\substack{i=1\\j=i}}^{n}\left(\left(\underbrace{\text{ķ}}\,\underbrace{\text{æ}}_{i}\oplus\underbrace{y}\,\underbrace{\text{æ}}_{j}\right)^{\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}}\right)=$$
$$\left\langle\sqrt{1-e^{-\left(\frac{1}{\underbrace{\text{ķ}}+\underbrace{y}}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\underbrace{\text{ķ}}\left(-\ln\left(1-\mu_i^2\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-\mu_j^2\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\,e^{2\pi\iota\sqrt{1-e^{-\left(\frac{1}{\underbrace{\text{ķ}}+\underbrace{y}}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{\underbrace{n}\left(\underbrace{n}+1\right)}\left(\underbrace{\text{ķ}}\left(-\ln\left(1-\alpha_i^2\right)\right)^{y}+\underbrace{y}\left(-\ln\left(1-\alpha_j^2\right)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}},\right.$$

$$\sqrt{e^{-\left(\frac{1}{k+y}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{n(n+1)}\left(k\left(-\ln(¥_i^2)\right)^{y}+y\left(-\ln(¥_j^2)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\; e^{2\pi\iota\sqrt{e^{-\left(\frac{1}{k+y}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{n(n+1)}\left(k\left(-\ln(\beta_i^2)\right)^{y}+y\left(-\ln(\beta_j^2)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}}\Bigg\rangle$$

Hence,

$$\mathrm{CPyFAAGHM}^{k,y}(æ_1,æ_2,\dots,æ_n)=$$

$$\Bigg\langle\sqrt{1-e^{-\left(\frac{1}{k+y}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{n(n+1)}\left(k\left(-\ln(1-\mu_i^2)\right)^{y}+y\left(-\ln(1-\mu_j^2)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\; e^{2\pi\iota\sqrt{1-e^{-\left(\frac{1}{k+y}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{n(n+1)}\left(k\left(-\ln(1-\alpha_i^2)\right)^{y}+y\left(-\ln(1-\alpha_j^2)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}},$$

$$\sqrt{e^{-\left(\frac{1}{k+y}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{n(n+1)}\left(k\left(-\ln(¥_i^2)\right)^{y}+y\left(-\ln(¥_j^2)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}\; e^{2\pi\iota\sqrt{e^{-\left(\frac{1}{k+y}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\frac{2}{n(n+1)}\left(k\left(-\ln(\beta_i^2)\right)^{y}+y\left(-\ln(\beta_i^2)\right)^{y}\right)\right)\right)\right)^{\frac{1}{y}}}}}\Bigg\rangle.$$

Definition 15 [62]. Let $æ_i = (\mu_{æ_i}(x)e^{2\pi\iota\left(\alpha_{æ_i}(x)\right)}, ¥_{æ_i}(x)e^{2\pi\iota\left(\beta_{æ_i}(x)\right)})$, as a family of real numbers, with weight vector $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_n)^T$, $\vartheta_n \in [0, 1]$, $\sum_{i=1}^{n} \vartheta_i = 1$ and $k > 0$, $y > 0$. Then, a CPyFAAWGHM is expressed as follows:

$$\mathrm{CPyFAAWGHM}^{k,y}(æ_1, æ_2, \dots, æ_n) = \frac{1}{k+y}\left(\prod_{\substack{i=1\\j=i}}^{n} k\left(æ_i\right)^{\vartheta_i} \oplus y\left(æ_j\right)^{\vartheta_j}\right)^{\frac{2}{n(n+1)}} \tag{12}$$

Theorem 12. Let $æ_i = (\mu_{æ_i}(x)e^{2\pi\iota\left(\alpha_{æ_i}(x)\right)}, ¥_{æ_i}(x)e^{2\pi\iota\left(\beta_{æ_i}(x)\right)})$, be a family of CPyFVs, with weight vector $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_n)^T$, $\vartheta_n \in [0, 1]$, $\sum_{i=1}^{n} \vartheta_i = 1$ and $k > 0$, $y > 0$. Then, a CPyFAAWGHM is defined as follows:

$$\mathrm{CPyFAAWGHM}^{k,y}(æ_1, æ_2, \dots, æ_n) =$$

$$\Bigg\langle\sqrt{1-e^{-\left(\left(\frac{1}{k+y}\right)\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(k\left(-\ln\left(1-e^{-\left(\vartheta_i\left(-\ln(\mu_i^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+y\left(-\ln\left(1-e^{-\left(\vartheta_j\left(-\ln(\mu_j^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}$$

$$e^{2\pi\iota\sqrt{1-e^{-\left(\left(\frac{1}{k+y}\right)\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(k\left(-\ln\left(1-e^{-\left(\vartheta_i\left(-\ln(\alpha_i^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+y\left(-\ln\left(1-e^{-\left(\vartheta_j\left(-\ln(\alpha_j^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}},$$

$$\sqrt{e^{-\left(\left(\frac{1}{k+y}\right)\left(\frac{2}{n(n+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(k\left(-\ln\left(1-e^{-\left(\vartheta_i\left(-\ln(1-¥_i^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+y\left(-\ln\left(1-e^{-\left(\vartheta_j\left(-\ln(1-¥_j^2)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}$$

$$e^{2\pi\iota\sqrt{e^{-\left(\left(\frac{1}{\underbrace{k}+\underbrace{y}}\right)\left(\frac{2}{\underbrace{n}(\underbrace{n}+1)}\right)\left(\sum_{\substack{i=1\\ j=i}}^{n}\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln\left(1-\beta_i^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y+\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_j\left(-\ln\left(1-\beta_j^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)\right)\right)^{\frac{1}{y}}}}}\Bigg\rangle.$$

***Proof*.**

Let $\underbrace{æ}_i=(\mu_{\underbrace{æ}_i}(x)e^{2\pi\iota\left(\alpha_{\underbrace{æ}_i}(x)\right)}, ¥_{\underbrace{æ}_i}(x)e^{2\pi\iota\left(\beta_{\underbrace{æ}_i}(x)\right)})$, i = 1, 2, …, n, we have

$$\left(\underbrace{æ}_i\right)^{\underbrace{\vartheta}_i}=$$

$$\left\langle\sqrt{e^{-\left(\underbrace{\vartheta}_i\left(-\ln\left(\mu_i^2\right)\right)^y\right)^{\frac{1}{y}}}}e^{2\pi\iota\sqrt{e^{-\left(\underbrace{\vartheta}_i\left(-\ln\left(\alpha_i^2\right)\right)^y\right)^{\frac{1}{y}}}}},\sqrt{1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln\left(1-¥_i^2\right)\right)^y\right)^{\frac{1}{y}}}}\ e^{2\pi\iota\sqrt{1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln\left(1-\beta_i^2\right)\right)^y\right)^{\frac{1}{y}}}}}\right\rangle$$

$$\left(\underbrace{æ}_j\right)^{\underbrace{\vartheta}_j}=$$

$$\left\langle\sqrt{e^{-\left(\underbrace{\vartheta}_j\left(-\ln\left(\mu_j^2\right)\right)^y\right)^{\frac{1}{y}}}}e^{2\pi\iota\sqrt{e^{-\left(\underbrace{\vartheta}_j\left(-\ln\left(\alpha_j^2\right)\right)^y\right)^{\frac{1}{y}}}}},\sqrt{1-e^{-\left(\underbrace{\vartheta}_j\left(-\ln\left(1-¥_j^2\right)\right)^y\right)^{\frac{1}{y}}}}\ e^{2\pi\iota\sqrt{1-e^{-\left(\underbrace{\vartheta}_j\left(-\ln\left(1-\beta_j^2\right)\right)^y\right)^{\frac{1}{y}}}}}\right\rangle$$

$$\underbrace{k}\left(\underbrace{æ}_i\right)^{\underbrace{\vartheta}_i}=\left\langle\sqrt{1-e^{-\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln\left(\mu_i^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)^{\frac{1}{y}}}}\ e^{2\pi\iota\sqrt{1-e^{-\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln\left(\alpha_i^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)^{\frac{1}{y}}}}},\right.$$

$$\left.\sqrt{e^{-\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln\left(1-¥_i^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)^{\frac{1}{y}}}}\ e^{2\pi\iota\sqrt{e^{-\left(\underbrace{k}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln\left(1-\beta_i^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)^{\frac{1}{y}}}}}\right\rangle$$

$$\underbrace{y}\left(\underbrace{æ}_j\right)^{\underbrace{\vartheta}_j}=\left\langle\sqrt{1-e^{-\left(\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_j\left(-\ln\left(\mu_j^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)^{\frac{1}{y}}}}\ e^{2\pi\iota\sqrt{1-e^{-\left(\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln\left(\alpha_j^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)^{\frac{1}{y}}}}},\right.$$

$$\left.\sqrt{e^{-\left(\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_j\left(-\ln\left(1-¥_j^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)^{\frac{1}{y}}}}\ e^{2\pi\iota\sqrt{e^{-\left(\underbrace{y}\left(-\ln\left(1-e^{-\left(\underbrace{\vartheta}_i\left(-\ln\left(1-\beta_j^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)^{\frac{1}{y}}}}}\right\rangle$$

$$\underset{\sim}{k}\left(\underset{\sim}{æ}_i\right)^{\underset{\sim}{\vartheta}_i} \oplus \underset{\sim}{y}\left(\underset{\sim}{æ}_j\right)^{\underset{\sim}{\vartheta}_j} = \left\langle \sqrt{1-e^{-\left(\underset{\sim}{k}\left(-\ln\left(1-e^{-\left(\underset{\sim}{\vartheta}_i\left(-\ln\left(\mu_i^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y+\underset{\sim}{y}\left(-\ln\left(1-e^{-\left(\underset{\sim}{\vartheta}_j\left(-\ln\left(\mu_j^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)^{\frac{1}{y}}}}\; e^{2\pi\iota\sqrt{1-e^{-\left(\underset{\sim}{k}\left(-\ln\left(1-e^{-\left(\underset{\sim}{\vartheta}_i\left(-\ln\left(\alpha_i^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y+\underset{\sim}{y}\left(-\ln\left(1-e^{-\left(\underset{\sim}{\vartheta}_j\left(-\ln\left(\alpha_j^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)^{\frac{1}{y}}}}},\right.$$

$$\left.\sqrt{e^{-\left(\underset{\sim}{k}\left(-\ln\left(1-e^{-\left(\underset{\sim}{\vartheta}_i\left(-\ln\left(1-\text{¥}_i^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y+\underset{\sim}{y}\left(-\ln\left(1-e^{-\left(\underset{\sim}{\vartheta}_j\left(-\ln\left(1-\text{¥}_j^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)^{\frac{1}{y}}}}\; e^{2\pi\iota\sqrt{e^{-\left(\underset{\sim}{k}\left(-\ln\left(1-e^{-\left(\underset{\sim}{\vartheta}_i\left(-\ln\left(1-\beta_i^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y+\underset{\sim}{y}\left(-\ln\left(1-e^{-\left(\vartheta_j\left(-\ln\left(1-\beta_j^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)^{\frac{1}{y}}}}}\right\rangle$$

Now let $z = 1-e^{-\left(\underset{\sim}{k}\left(-\ln\left(1-e^{-\left(\underset{\sim}{\vartheta}_i\left(-\ln\left(\mu_i^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y+\underset{\sim}{y}\left(-\ln\left(1-e^{-\left(\underset{\sim}{\vartheta}_j\left(-\ln\left(\mu_j^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)^{\frac{1}{y}}}$

$$\ln(1-z) = -\left(\underset{\sim}{k}\left(-\ln\left(1-e^{-\left(\underset{\sim}{\vartheta}_i\left(-\ln\left(\mu_i^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y+\underset{\sim}{y}\left(-\ln\left(1-e^{-\left(\underset{\sim}{\vartheta}_j\left(-\ln\left(\mu_j^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)^{\frac{1}{y}}$$

Utilizing this we get,

$$\prod_{\substack{i=1\\j=i}}^{n}\underset{\sim}{k}\left(\underset{\sim}{æ}_i\right)^{\underset{\sim}{\vartheta}_i} \oplus \underset{\sim}{y}\left(\underset{\sim}{æ}_j\right)^{\underset{\sim}{\vartheta}_j} = \left\langle \sqrt{e^{-\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\sim}{k}\left(-\ln\left(1-e^{-\left(\underset{\sim}{\vartheta}_i\left(-\ln\left(\mu_i^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y+\underset{\sim}{y}\left(-\ln\left(1-e^{-\left(\underset{\sim}{\vartheta}_j\left(-\ln\left(\mu_j^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)\right)^{\frac{1}{y}}}}\right.$$

$$e^{2\pi\iota\sqrt{e^{-\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\sim}{k}\left(-\ln\left(1-e^{-\left(\underset{\sim}{\vartheta}_i\left(-\ln\left(\alpha_i^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y+\underset{\sim}{y}\left(-\ln\left(1-e^{-\left(\underset{\sim}{\vartheta}_j\left(-\ln\left(\alpha_j^2\right)\right)^y\right)^{\frac{1}{y}}}\right)\right)^y\right)\right)^{\frac{1}{y}}}}},$$

$$\sqrt{1-e^{-\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(1-\yen_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(1-\yen_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)^{\frac{1}{y}}}}$$

$$e^{2\pi\iota\sqrt{1-e^{-\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(1-\beta_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(1-\beta_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)^{\frac{1}{y}}}}}\Bigg\rangle$$

$$\left(\prod_{\substack{i=1\\j=i}}^{n}\underset{\smile}{k}\left(\underset{\smile}{æ}_i\right)^{\underset{\smile}{\vartheta}_i}\oplus\underset{\smile}{y}\left(\underset{\smile}{æ}_j\right)^{\underset{\smile}{\vartheta}_j}\right)^{\frac{2}{\underset{\smile}{n}\left(\underset{\smile}{n}+1\right)}}=$$

$$\Bigg\langle\sqrt{e^{-\left(\frac{2}{\underset{\smile}{n}\left(\underset{\smile}{n}+1\right)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(\mu_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(\mu_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}$$

$$e^{2\pi\iota\sqrt{e^{-\left(\frac{2}{\underset{\smile}{n}\left(\underset{\smile}{n}+1\right)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(\alpha_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(\alpha_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}},$$

$$\sqrt{1-e^{-\left(\frac{2}{\underset{\smile}{n}\left(\underset{\smile}{n}+1\right)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(1-\yen_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(1-\yen_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}$$

$$e^{2\pi\iota\sqrt{1-e^{-\left(\frac{2}{\underset{\smile}{n}\left(\underset{\smile}{n}+1\right)}\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(1-\beta_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(1-\beta_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}}\Bigg\rangle$$

$$\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\left(\prod_{\substack{i=1\\j=i}}^{n}\underset{\smile}{k}\left(\underset{\smile}{æ}_i\right)^{\underset{\smile}{\vartheta}_i}\oplus\underset{\smile}{y}\left(\underset{\smile}{æ}_j\right)^{\underset{\smile}{\vartheta}_j}\right)^{\frac{2}{\underset{\smile}{n}\left(\underset{\smile}{n}+1\right)}}=$$

$$\Bigg\langle\sqrt{1-e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\frac{2}{\underset{\smile}{n}\left(\underset{\smile}{n}+1\right)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(\mu_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(\mu_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}$$

$$e^{2\pi\iota\sqrt{1-e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(\alpha_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(\alpha_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}},$$

$$\sqrt{e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(1-¥_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(1-¥_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}},$$

$$e^{2\pi\iota\sqrt{e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(1-\beta_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(1-\beta_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}}\Bigg\rangle$$

Hence,

$CPyFAAWGHM^{\underset{\smile}{k},\underset{\smile}{y}}(\underset{\smile}{æ}_1, \underset{\smile}{æ}_2, \dots, \underset{\smile}{æ}_n)=$

$$\Bigg\langle\sqrt{1-e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(\mu_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(\mu_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}$$

$$e^{2\pi\iota\sqrt{1-e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(\alpha_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(\alpha_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}},$$

$$\sqrt{e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(1-¥_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(1-¥_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}$$

$$e^{2\pi\iota\sqrt{e^{-\left(\left(\frac{1}{\underset{\smile}{k}+\underset{\smile}{y}}\right)\left(\frac{2}{\underset{\smile}{n}(\underset{\smile}{n}+1)}\right)\left(\sum_{\substack{i=1\\j=i}}^{n}\left(\underset{\smile}{k}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_i\left(-\ln\left(1-\beta_i^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}+\underset{\smile}{y}\left(-\ln\left(1-e^{-\left(\underset{\smile}{\vartheta}_j\left(-\ln\left(1-\beta_j^2\right)\right)^{y}\right)^{\frac{1}{y}}}\right)\right)^{y}\right)\right)\right)^{\frac{1}{y}}}}}\Bigg\rangle.$$

6. MCDM Techniques Based on Complex Pythagorean Fuzzy Sets

The process of picking an acceptable substitute from an array of possibilities utilizing characteristics that the decision-maker established is referred to as a MCDM the strategy. Imagine of a definite set of alternatives, $\Psi=(\Psi_1,\Psi_2,\ldots,\Psi_n)$, where the decision-maker's defined features are depicted by $\varsigma=(\varsigma_1,\varsigma_2,\ldots,\varsigma_n)$. The set of constrained alternatives in the context of CPyFVs that are based on features closed in [0, 1]. The set of weight vectors is marked by $\wp=(\wp_1, \wp_2,\ldots, \wp_n)$, $\wp_i \in [0, 1]$, and $\sum_{i=1}^{n} \wp_i = 1, i = (1, 2, 3,\ldots,n)$. Analyze the choice matrix $\underbrace{æ} = (\mu(x)e^{2\pi i(\alpha(x))}, ¥(x)e^{2\pi i(\beta(x))})\rho\times\varphi$ that decision-maker supplied. The CPyFVs are symbolized by each pair $(\mu_{\rho\varphi}e^{2\pi i(\alpha_{\rho\varphi}(x))}, ¥_{\rho\varphi}e^{2\pi i(\beta_{\rho\varphi}(x))})$, which must comply with the prerequisite that $0 \le \mu^2 + ¥^2 \le 1$ and $0 \le \alpha^2 + \beta^2 \le 1$.

$$\underbrace{æ}_i = \left(\mu_{\underbrace{æ}_i}(x)e^{2\pi\iota\left(\alpha_{\underbrace{æ}_i}(x)\right)}, ¥_{\underbrace{æ}_i}(x)e^{2\pi\iota\left(\beta_{\underbrace{æ}_i}(x)\right)}\right)\rho \times \varphi$$

$$= \begin{bmatrix} \left(\mu_{11}e^{2\pi i(\alpha_{11}(x))}, ¥_{11}e^{2\pi i(\beta_{11}(x))}\right) & \left(\mu_{11}e^{2\pi i(\alpha_{11}(x))}, ¥_{11}e^{2\pi i(\beta_{11}(x))}\right) & \ldots & \left(\mu_{1\varphi}e^{2\pi i(\alpha_{1\varphi}(x))}, ¥_{1\varphi}e^{2\pi i(\beta_{1\varphi}(x))}\right) \\ \left(\mu_{21}e^{2\pi i(\alpha_{11}(x))}, ¥_{21}e^{2\pi i(\beta_{21}(x))}\right) & \left(\mu_{22}e^{2\pi i(\alpha_{22}(x))}, ¥_{22}e^{2\pi i(\beta_{22}(x))}\right) & \ldots & \left(\mu_{2\varphi}e^{2\pi i(\alpha_{2\varphi}(x))}, ¥_{2\varphi}e^{2\pi i(\beta_{2\varphi}(x))}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\mu_{\rho1}e^{2\pi i(\alpha_{11}(x))}, ¥_{\rho1}e^{2\pi i(\beta_{\rho1}(x))}\right) & \left(\mu_{\rho2}e^{2\pi i(\alpha_{\rho2}(x))}, ¥_{\rho2}e^{2\pi i(\beta_{\rho2}(x))}\right) & \ldots & \left(\mu_{\rho\varphi}e^{2\pi i(\alpha_{\rho\varphi}(x))}, ¥_{\rho\varphi}e^{2\pi i(\beta_{\rho\varphi}(x))}\right) \end{bmatrix}$$

Using AOs of CPyFAAWHM as well as CPyFAAWGHM operators in regard to MCDM approach in context of the CPyF dossier, we demonstrate the best option. Using the algorithm's subsequent stages, we assessed complex Pythagorean fuzzy data under a MCDM approach. We also looked at the algorithm further phases in the Figure 1 follow chart.

Step 1. The decision-maker gathers data in the form of CPyFVs and displays all available data as decision matrices.

Step 2. The supplied decision matrix $\underbrace{æ}_i = (\mu_{\underbrace{æ}_i}(x)e^{2\pi\iota\left(\alpha_{\underbrace{æ}_i}(x)\right)}, ¥_{\underbrace{æ}_i}(x)e^{2\pi\iota\left(\beta_{\underbrace{æ}_i}(x)\right)})\rho\times\varphi$ has to be transformed into the normalization matrix $\underbrace{æ}_i = (\mu_{\underbrace{æ}_i}(x)e^{2\pi\iota\left(\alpha_{\underbrace{æ}_i}(x)\right)}, ¥_{\underbrace{æ}_i}(x)e^{2\pi\iota\left(\beta_{\underbrace{æ}_i}(x)\right)})\rho\times\varphi$. If the collection of characteristics encompasses numerous attribute sorts, such as cost and benefit types. The decision matrix does not need to be modified or changed into a normalization matrix in any other scenario.

Step 3. Based on certain features, we applied our strong suggested approaches, namely CPyFAAWHM along with CPyFAAWGHM operators, to identify the best choice $\Psi=(\Psi_1, \Psi_2, \ldots, \Psi_n)$.

Step 4. Use the acquired CPyFAAWHM and CPyFAAWGHM operator consequences to compute score values.

$$\bar{\bar{C}}(\underbrace{æ}) = \frac{(\mu(x))^2 - \left(¥(x)\right)^2 + (\alpha(x))^2 - (\beta(x))^2}{2},$$

Where $\bar{\bar{C}}(\underbrace{æ}) \in [-1, 1]$.

Step 5. After the score values have been established, reorder the score values utilizing the ranking and ordering method to look for more suitable competitors.

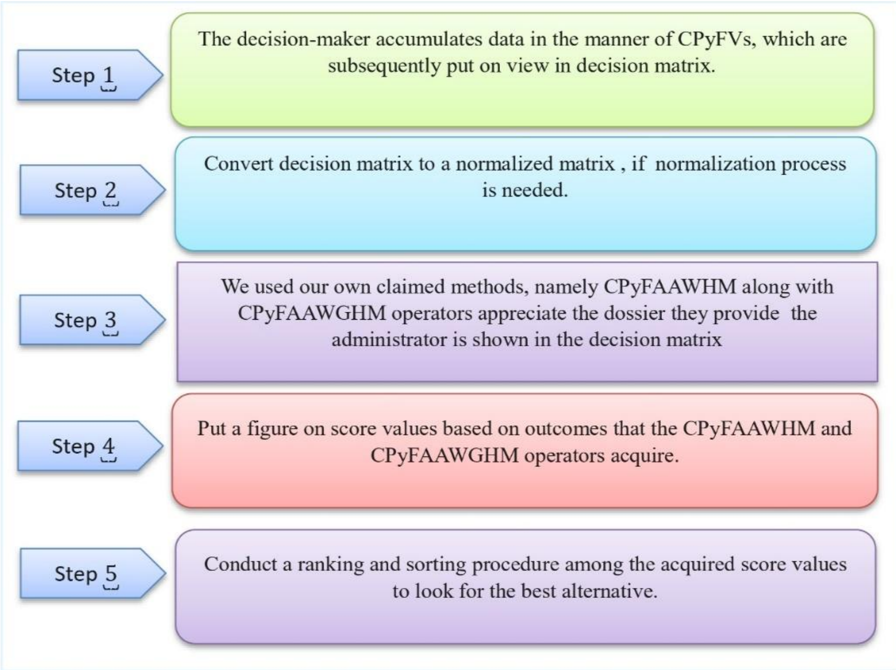


Fig. 1. Flow chart of complex Pythagorean fuzzy data under a MCDM approach

6.1. Numerical Example

An enterprises company needs to hire a new boss. There are five people applying for the job. Contemplate $\bar{\bar{\rho}}_n$, n= (1, 2, 3, 4, 5) be the consignment of five distinctive candidates. The people in charge want to finish choosing based on following qualities. $\underbrace{C}_1$ represents the Applicants' qualifications, $\underbrace{C}_2$ is a way to show what the applicants have done. $\underbrace{C}_3$ stand for the way people act , $\underbrace{C}_4$ shows personality shown by the applicants and $\underbrace{C}_5$ represents applicants' s behavior. Where $\underbrace{C}_1$ and $\underbrace{C}_5$ are cost type while $\underbrace{C}_2$, $\underbrace{C}_3$ and $\underbrace{C}_4$ are benefit type. The analogous weight vectors $\underbrace{\vartheta}$ = (0.2, 0.1, 0.3, 0.25, 0.15). The person in charge shared some information shown in Table 1 as CPyFVs.

Table 1
Decision Matrix

	$\underbrace{C}_1$	$\underbrace{C}_2$	$\underbrace{C}_3$	$\underbrace{C}_4$	$\underbrace{C}_5$
$\bar{\bar{\rho}}_1$	$(0.2e^{2\pi\iota(0.55)}, 0.32e^{2\pi i(0.46)})$	$(0.36e^{2\pi\iota(0.82)}, 0.45e^{2\pi\iota(0.43)})$	$(0.46e^{2\pi\iota(0.27)}, 0.7e^{2\pi\iota(0.61)})$	$(0.7e^{2\pi\iota(0.57)}, 0.36e^{2\pi\iota(0.38)})$	$(0.55e^{2\pi\iota(0.88)}, 0.46e^{2\pi\iota(0.45)})$
$\bar{\bar{\rho}}_2$	$(0.42e^{2\pi\iota(0.46)}, 0.03e^{2\pi\iota(0.09)})$	$(0.09e^{2\pi\iota(0.39)}, 0.46e^{2\pi\iota(0.35)})$	$(0.59e^{2\pi\iota(0.33)}, 0.4e^{2\pi\iota(0.41)})$	$(0.24e^{2\pi\iota(0.59)}, 0.03e^{2\pi\iota(0.4)})$	$(0.48e^{2\pi\iota(0.37)}, 0.56e^{2\pi\iota(0.67)})$
$\bar{\bar{\rho}}_3$	$(0.5e^{2\pi\iota(0.15)}, 0.24e^{2\pi\iota(0.08)})$	$(0.78e^{2\pi\iota(0.55)}, 0.28e^{2\pi\iota(0.18)})$	$(0.34e^{2\pi\iota(0.78)}, 0.62e^{2\pi\iota(0.28)})$	$(0.4e^{2\pi\iota(0.38)}, 0.24e^{2\pi\iota(0.75)})$	$(0.29e^{2\pi\iota(0.81)}, 0.71e^{2\pi\iota(0.09)})$
$\bar{\bar{\rho}}_4$	$(0.34e^{2\pi\iota(0.36)}, 0.02e^{2\pi\iota(0.06)})$	$(0.48e^{2\pi\iota(0.47)}, 0.45e^{2\pi\iota(0.42)})$	$(0.65e^{2\pi\iota(0.57)}, 0.19e^{2\pi\iota(0.38)})$	$(0.62e^{2\pi\iota(0.36)}, 0.02e^{2\pi\iota(0.55)})$	$(0.33e^{2\pi\iota(0.27)}, 0.41e^{2\pi\iota(0.61)})$
$\bar{\bar{\rho}}_5$	$(0.043e^{2\pi\iota(0.48)}, 0.56e^{2\pi\iota(0.19)})$	$(0.65e^{2\pi\iota(0.57)}, 0.25e^{2\pi\iota(0.38)})$	$(0.45e^{2\pi\iota(0.46)}, 0.01e^{2\pi\iota(0.47)})$	$(0.39e^{2\pi\iota(0.8)}, 0.65e^{2\pi\iota(0.53)})$	$(0.28e^{2\pi\iota(0.12)}, 0.14e^{2\pi\iota(0.05)})$

6.2. The Method for Assessing the MCDM Approach

Step 1. The dossier regarding applicants, seen in Table 1 and is expected by administrator to be in manner of CPyFVs.

Step 2. We can observe that $\underbrace{C}_{i}$, i = 1, 2, 3,4 and $\underbrace{C}_{5}$ are benefit and cost types, respectively, as there are two categories of attributes that correspond to these types. Following the conversion of Table 1 conventional decision matrix into Table 2 normalizer matrix.

Table 2
Normalized Decision Matrix

	$\underbrace{C}_{1}$	$\underbrace{C}_{2}$	$\underbrace{C}_{3}$	$\underbrace{C}_{4}$	$\underbrace{C}_{5}$
$\bar{\bar{\rho}}_1$	$(O.\ 32e^{2\pi i(0.46)}, 0.2e^{2\pi i(0.55)})$	$(0.36e^{2\pi i(0.82)}, 0.45e^{2\pi i(0.43)})$	$(0.46e^{2\pi i(0.27)}, 0.7e^{2\pi i(0.61)})$	$(0.7e^{2\pi i(0.57)}, 0.36e^{2\pi i(0.38)})$	$(0.46e^{2\pi i(0.45)}, 0.55e^{2\pi i(0.88)})$
$\bar{\bar{\rho}}_2$	$(0.03e^{2\pi i(0.09)}, 0.42e^{2\pi i(0.46)})$	$(0.09e^{2\pi i(0.39)}, 0.46e^{2\pi i(0.35)})$	$(0.59e^{2\pi i(0.33)}, 0.4e^{2\pi i(0.41)})$	$(0.24e^{2\pi i(0.59)}, 0.03e^{2\pi i(0.4)})$	$(0.56e^{2\pi i(0.67)}, 0.48e^{2\pi i(0.37)})$
$\bar{\bar{\rho}}_3$	$(0.24e^{2\pi i(0.08)}, O.5e^{2\pi i(0.15)})$	$(0.78e^{2\pi i(0.55)}, 0.28e^{2\pi i(0.18)})$	$(0.34e^{2\pi i(0.78)}, 0.62e^{2\pi i(0.28)})$	$(0.4e^{2\pi i(0.38)}, 0.24e^{2\pi i(0.75)})$	$(0.71e^{2\pi i(0.09)}, 0.29e^{2\pi i(0.81)})$
$\bar{\bar{\rho}}_4$	$(0.02e^{2\pi i(0.06)}, 0.34e^{2\pi i(0.36)})$	$(0.48e^{2\pi i(0.47)}, 0.45e^{2\pi i(0.42)})$	$(0.65e^{2\pi i(0.57)}, 0.19e^{2\pi i(0.38)})$	$(0.62e^{2\pi i(0.36)}, 0.02e^{2\pi i(0.55)})$	$(0.41e^{2\pi i(0.61)}, 0.33e^{2\pi i(0.27)})$
$\bar{\bar{\rho}}_5$	$(0.56e^{2\pi i(0.19)}, 0.043e^{2\pi i(0.48)})$	$(0.65e^{2\pi i(0.57)}, 0.25e^{2\pi i(0.38)})$	$(0.45e^{2\pi i(0.46)}, 0.01e^{2\pi i(0.47)})$	$(0.39e^{2\pi i(0.8)}, 0.65e^{2\pi i(0.53)})$	$(0.14e^{2\pi i(0.05)}, 0.28e^{2\pi i(0.12)})$

Step 3. Using the CPyFAAWGHM and CPyFAAWHM operators, we applied our suggested techniques to Table 2 of the decision matrix. Table 3 shows all of the results that were obtained using the suggested procedures for У= 9.

Table 3
Consequence of CPyFAAWHM along with CPyFAAWGHM operators at У = 9

τ_i	CPyFAAWHM	CPyFAAWGHM
τ_1	$(0.03041e^{2\pi i(0.0758)}, 0.5102e^{2\pi i(0.56347)})$	$(0.58917e^{2\pi i(0.47581)}, 0.061e^{2\pi i(0.19338)})$
τ_2	$(0.12439e^{2\pi i(0.41185)}, 0.51315e^{2\pi i(0.46704)})$	$(0.76358e^{2\pi i(0.79929)}, 0.26723e^{2\pi i(0.22423)})$
τ_3	$(0.37826e^{2\pi i(0.32714)}, 0.69096e^{2\pi i(0.60961)})$	$(0.65787e^{2\pi i(0.76281)}, 0.01741e^{2\pi i(0.31916)})$
τ_4	$(0.277e^{2\pi i(0.37821)}, 0.66844e^{2\pi i(0.73469)})$	$(0.69161e^{2\pi i(0.80096)}, 0.03041e^{2\pi i(0.40824)})$
τ_5	$(0.16851e^{2\pi i(0.0675)}, 0.56175e^{2\pi i(0.8587)})$	$(0.6979e^{2\pi i(0.68737)}, 0.28507e^{2\pi i(0.14716)})$

Step 4. Using Definition 3, we calculated the score values. Table 4 displays the outcomes of the investigation into the score values.

Table 4
Score values of CPyFAAWHM along with CPyFAAWGHM operators

Operators	$\bar{\bar{C}}(\bar{\bar{\rho}}_1)$	$\bar{\bar{C}}(\overline{\overline{\rho_2}})$	$\bar{\bar{C}}(\overline{\overline{\rho_3}})$	$\bar{\bar{C}}(\overline{\overline{\rho_4}})$	$\bar{\bar{C}}(\overline{\overline{\rho_5}})$	Ranking and ordering
CPyFAAWHM	-0.2856	-0.14818	-0.29947	-0.3834	-0.50999	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_1 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_5$
CPyFAAWGHM	0.265452	0.550114	0.456253	0.476138	0.428366	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_5) > \bar{\bar{\rho}}_1$

Step 5. We rank and order the options in order to look into applicant as a more acceptable option. Upon analyzing the score values, we find that $\underbrace{C}_{2}$ is a good substitute for the CPyFAAWGHM as well as CPyFAAWHM operators. Figure 2 shows the aftermath of the score values for the CPyFAAWGM as well as CPyFAAWHM operators as nomograph.

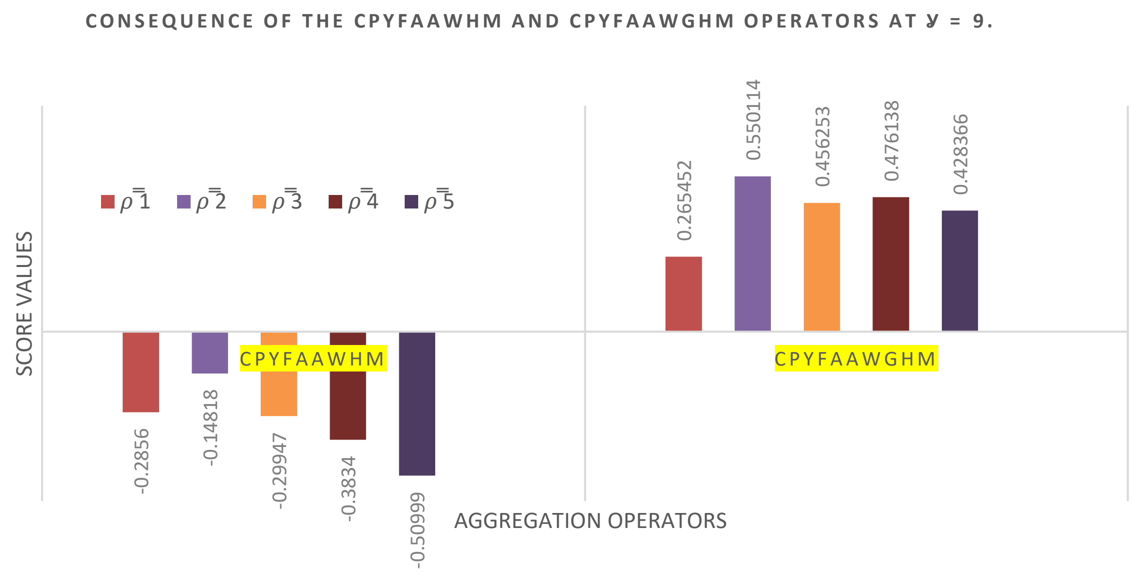


Fig. 2. Graph of the results of CPyFAAWHMO along with CPyFAAWGHMO

6.3. The influence of parameters У, $\underset{\smile}{\mathrm{k}}$ and $\underset{\smile}{\mathrm{y}}$ variation on our suggested approach

In this section, we analyze the influence of the parameters У, $\underset{\smile}{\mathrm{k}}$ and $\underset{\smile}{\mathrm{y}}$ on the performance and stability of the proposed aggregation operators. Sensitivity analysis is conducted to examine how variations in these parameters affect the computed score values and the resulting ranking of alternatives. Firstly, the effect of parameter У is investigated. Tables 5 and 6, along with Figures 3 and 5, illustrate the variation in score values obtained by the CPyFAAWHM and CPyFAAWGHM operators, respectively, for different values of У.

Table 5
Consequence of the score values of the CPyFAAWHM by the variation of У

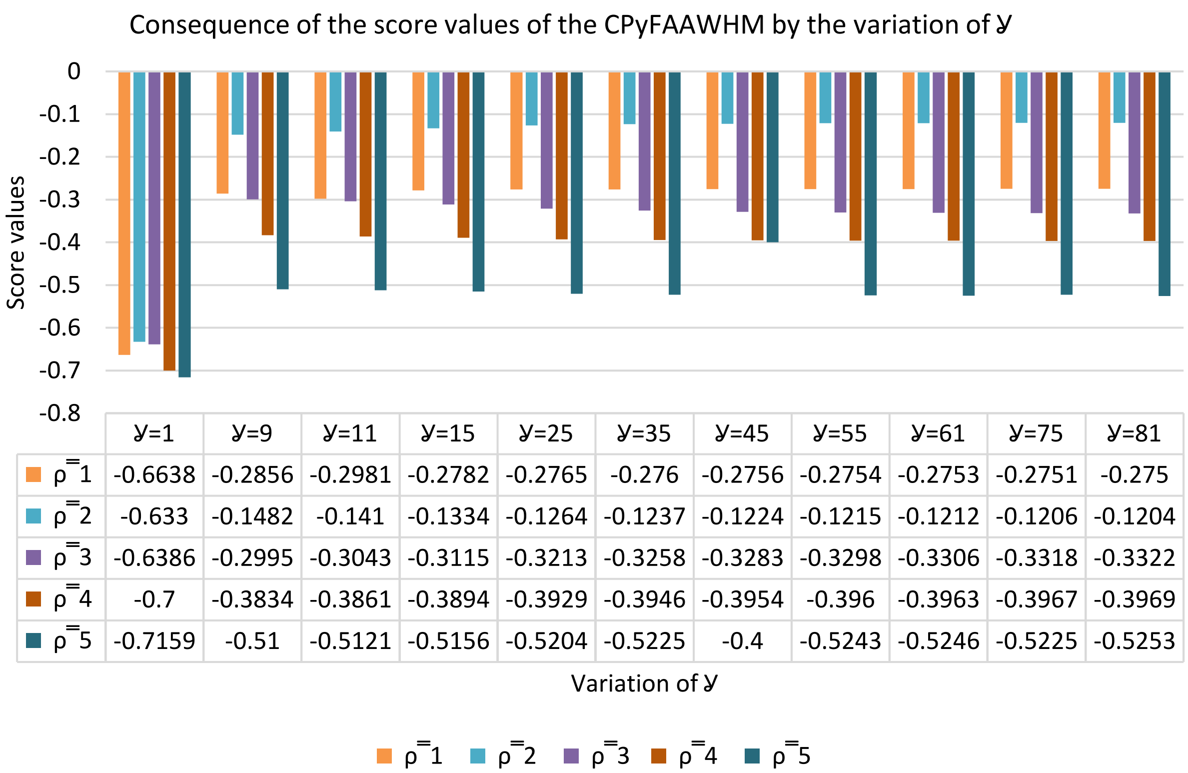
	$\bar{\bar{C}}(\overline{\overline{\rho_1}})$	$\bar{\bar{C}}(\overline{\overline{\rho_2}})$	$\bar{\bar{C}}(\overline{\overline{\rho_3}})$	$\bar{\bar{C}}(\overline{\overline{\rho_4}})$	$\bar{\bar{C}}(\overline{\overline{\rho_5}})$	Ranking and ordering
У=1	-0.66375	-0.63299	-0.63856	-0.70000	-0.71593	$\bar{\bar{\rho}}_2>\bar{\bar{\rho}}_3>\bar{\bar{\rho}}_1>\bar{\bar{\rho}}_4>\bar{\bar{\rho}}_5$
У=9	-0.2856	-0.14818	-0.29947	-0.3834	-0.50999	$\bar{\bar{\rho}}_2>\bar{\bar{\rho}}_1>\bar{\bar{\rho}}_3>\bar{\bar{\rho}}_4>\bar{\bar{\rho}}_5$
У=11	-0.29812	-0.14096	-0.3043	-0.38608	-0.51205	$\bar{\bar{\rho}}_2>\bar{\bar{\rho}}_1>\bar{\bar{\rho}}_3>\bar{\bar{\rho}}_4>\bar{\bar{\rho}}_5$
У=15	-0.27818	-0.13342	-0.31146	-0.38938	-0.51555	$\bar{\bar{\rho}}_2>\bar{\bar{\rho}}_1>\bar{\bar{\rho}}_3>\bar{\bar{\rho}}_4>\bar{\bar{\rho}}_5$
У=25	-0.2765	-0.12644	-0.32126	-0.39294	-0.52038	$\bar{\bar{\rho}}_2>\bar{\bar{\rho}}_1>\bar{\bar{\rho}}_3>\bar{\bar{\rho}}_4>\bar{\bar{\rho}}_5$
У=35	-0.27597	-0.12373	-0.32581	-0.39455	-0.52254	$\bar{\bar{\rho}}_2>\bar{\bar{\rho}}_1>\bar{\bar{\rho}}_3>\bar{\bar{\rho}}_4>\bar{\bar{\rho}}_5$
У=45	-0.27559	-0.12237	-0.32832	-0.39538	-0.40000	$\bar{\bar{\rho}}_2>\bar{\bar{\rho}}_1>\bar{\bar{\rho}}_3>\bar{\bar{\rho}}_4>\bar{\bar{\rho}}_5$
У=55	-0.27537	-0.12152	-0.32979	-0.39595	-0.52434	$\bar{\bar{\rho}}_2>\bar{\bar{\rho}}_1>\bar{\bar{\rho}}_3>\bar{\bar{\rho}}_4>\bar{\bar{\rho}}_5$
У=61	-0.27527	-0.12117	-0.33061	-0.39625	-0.52464	$\bar{\bar{\rho}}_2>\bar{\bar{\rho}}_1>\bar{\bar{\rho}}_3>\bar{\bar{\rho}}_4>\bar{\bar{\rho}}_5$
У=75	-0.27509	-0.12059	-0.3318	-0.39669	-0.52251	$\bar{\bar{\rho}}_2>\bar{\bar{\rho}}_1>\bar{\bar{\rho}}_3>\bar{\bar{\rho}}_4>\bar{\bar{\rho}}_5$
У=81	-0.27504	-0.1204	-0.33218	-0.3969	-0.52534	$\bar{\bar{\rho}}_2>\bar{\bar{\rho}}_1>\bar{\bar{\rho}}_3>\bar{\bar{\rho}}_4>\bar{\bar{\rho}}_5$

It can be observed that changing У leads to noticeable variations in the score values of the alternatives. However, a detailed analysis of Table 5 reveals that a slight change in the ranking order occurs when У increases from 1 to 9, particularly due to the interchange in the positions of $\bar{\bar{\rho}}_1$ and $\bar{\bar{\rho}}_3$. For all higher values (i.e., У $\geq$ 9), the ranking order remains unchanged. Importantly, the best alternative $\bar{\bar{\rho}}_2$ and the worst alternative $\bar{\bar{\rho}}_5$ remain consistent across all values of У, demonstrating the robustness of the CPyFAAWHM operator. Similarly, Table 6 shows that although the score values obtained using the CPyFAAWGHM operator vary with changes in У, the ranking order remains stable for У ≥ 9. This consistency further supports the reliability of the proposed approach in identifying the optimal alternative under varying parameter conditions.

Table 6
Consequence of the score values of the CPyFAAWGHM by the variation of У

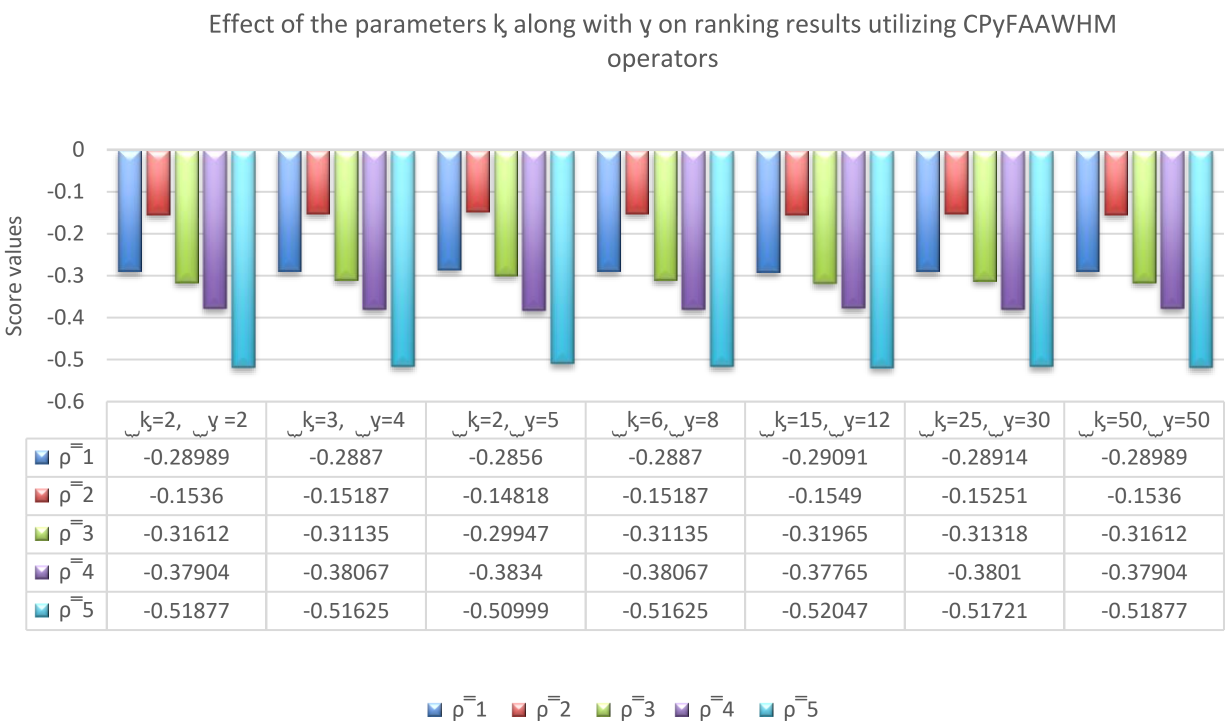
	$\bar{\bar{C}}(\bar{\bar{\rho}}_1)$	$\bar{\bar{C}}(\bar{\bar{\rho}}_2)$	$\bar{\bar{C}}(\bar{\bar{\rho}}_3)$	$\bar{\bar{C}}(\bar{\bar{\rho}}_4)$	$\bar{\bar{C}}(\bar{\bar{\rho}}_5)$	Ranking and ordering
У=1	0.525861	0.769428	0.75756	0.578764	0.686564	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_1$
У=9	0.265452	0.550114	0.456253	0.476138	0.428366	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_1$
У=11	0.264126	0.557545	0.458151	0.477751	0.428242	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_1$
У=15	0.261181	0.56691	0.461804	0.480392	0.428466	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_1$
У=25	0.257345	0.575109	0.464169	0.483908	0.428959	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_1$
У=35	0.255514	0.581771	0.469943	0.485649	0.429208	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_1$
У=45	0.254438	0.584259	0.473169	0.486776	0.429356	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_1$
У=55	0.25374	0.585858	0.472269	0.487617	0.429456	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_1$
У=61	0.253424	0.586561	0.472665	0.488034	0.4295	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_1$
У=75	0.252885	0.587762	0.473336	0.488795	0.42959	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_1$
У=81	0.252707	0.588151	0.473607	0.489058	0.429612	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_1$

Next, the influence of parameters $\underbrace{k}$ and $\underbrace{y}$ is examined, as presented in Figures 4 and 6. The results indicate that variations in these parameters lead to changes in the score values for both CPyFAAWHM and CPyFAAWGHM operators. However, despite these numerical differences, the overall ranking of alternatives remains unchanged. This demonstrates that the proposed operators are not significantly sensitive to the selection of $\underbrace{k}$ and $\underbrace{y}$ in terms of decision outcomes.



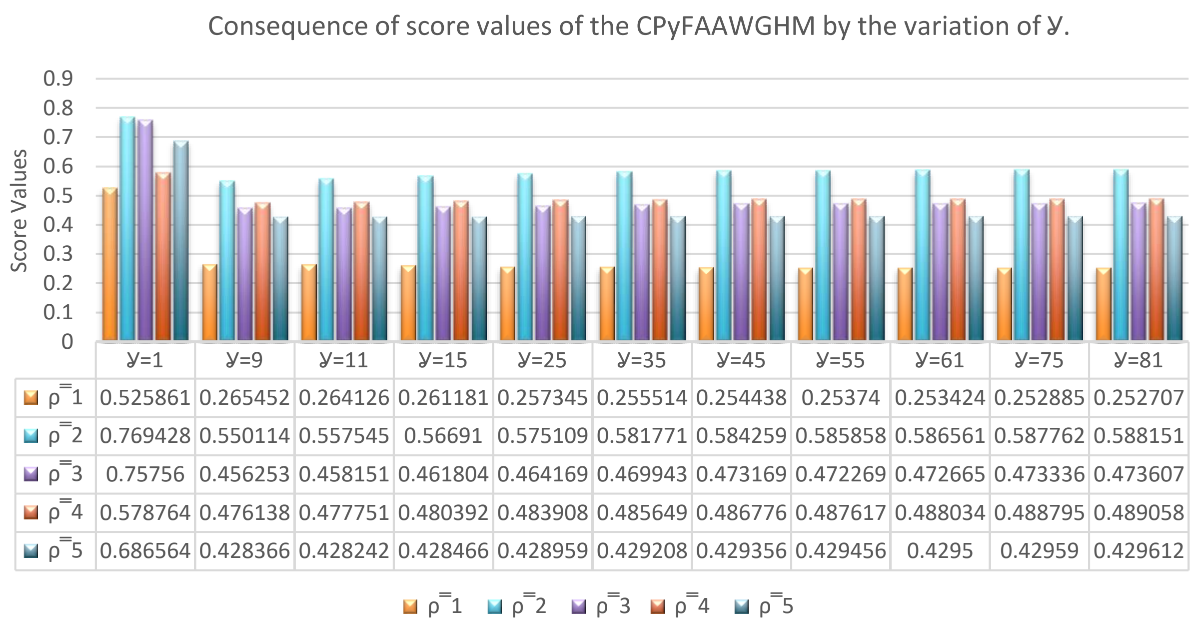
	У=1	У=9	У=11	У=15	У=25	У=35	У=45	У=55	У=61	У=75	У=81
$\bar{\bar{\rho}}1$	-0.6638	-0.2856	-0.2981	-0.2782	-0.2765	-0.276	-0.2756	-0.2754	-0.2753	-0.2751	-0.275
$\bar{\bar{\rho}}2$	-0.633	-0.1482	-0.141	-0.1334	-0.1264	-0.1237	-0.1224	-0.1215	-0.1212	-0.1206	-0.1204
$\bar{\bar{\rho}}3$	-0.6386	-0.2995	-0.3043	-0.3115	-0.3213	-0.3258	-0.3283	-0.3298	-0.3306	-0.3318	-0.3322
$\bar{\bar{\rho}}4$	-0.7	-0.3834	-0.3861	-0.3894	-0.3929	-0.3946	-0.3954	-0.396	-0.3963	-0.3967	-0.3969
$\bar{\bar{\rho}}5$	-0.7159	-0.51	-0.5121	-0.5156	-0.5204	-0.5225	-0.4	-0.5243	-0.5246	-0.5225	-0.5253

Fig. 3. Graph of the results of the CPyFAAWHM by the variation of У



	ķ=2, ỵ =2	ķ=3, ỵ=4	ķ=2, ỵ=5	ķ=6, ỵ=8	ķ=15, ỵ=12	ķ=25, ỵ=30	ķ=50, ỵ=50
ρ̿1	-0.28989	-0.2887	-0.2856	-0.2887	-0.29091	-0.28914	-0.28989
ρ̿2	-0.1536	-0.15187	-0.14818	-0.15187	-0.1549	-0.15251	-0.1536
ρ̿3	-0.31612	-0.31135	-0.29947	-0.31135	-0.31965	-0.31318	-0.31612
ρ̿4	-0.37904	-0.38067	-0.3834	-0.38067	-0.37765	-0.3801	-0.37904
ρ̿5	-0.51877	-0.51625	-0.50999	-0.51625	-0.52047	-0.51721	-0.51877

Fig. 4. Graph of the consequence of the parameters ķ and ỵ on ranking results of CPyFAAWHMOs



	Ұ=1	Ұ=9	Ұ=11	Ұ=15	Ұ=25	Ұ=35	Ұ=45	Ұ=55	Ұ=61	Ұ=75	Ұ=81
ρ̿1	0.525861	0.265452	0.264126	0.261181	0.257345	0.255514	0.254438	0.25374	0.253424	0.252885	0.252707
ρ̿2	0.769428	0.550114	0.557545	0.56691	0.575109	0.581771	0.584259	0.585858	0.586561	0.587762	0.588151
ρ̿3	0.75756	0.456253	0.458151	0.461804	0.464169	0.469943	0.473169	0.472269	0.472665	0.473336	0.473607
ρ̿4	0.578764	0.476138	0.477751	0.480392	0.483908	0.485649	0.486776	0.487617	0.488034	0.488795	0.489058
ρ̿5	0.686564	0.428366	0.428242	0.428466	0.428959	0.429208	0.429356	0.429456	0.4295	0.42959	0.429612

Fig. 5. Graph of the results of CPyFAAWGHM by the variation of Ұ

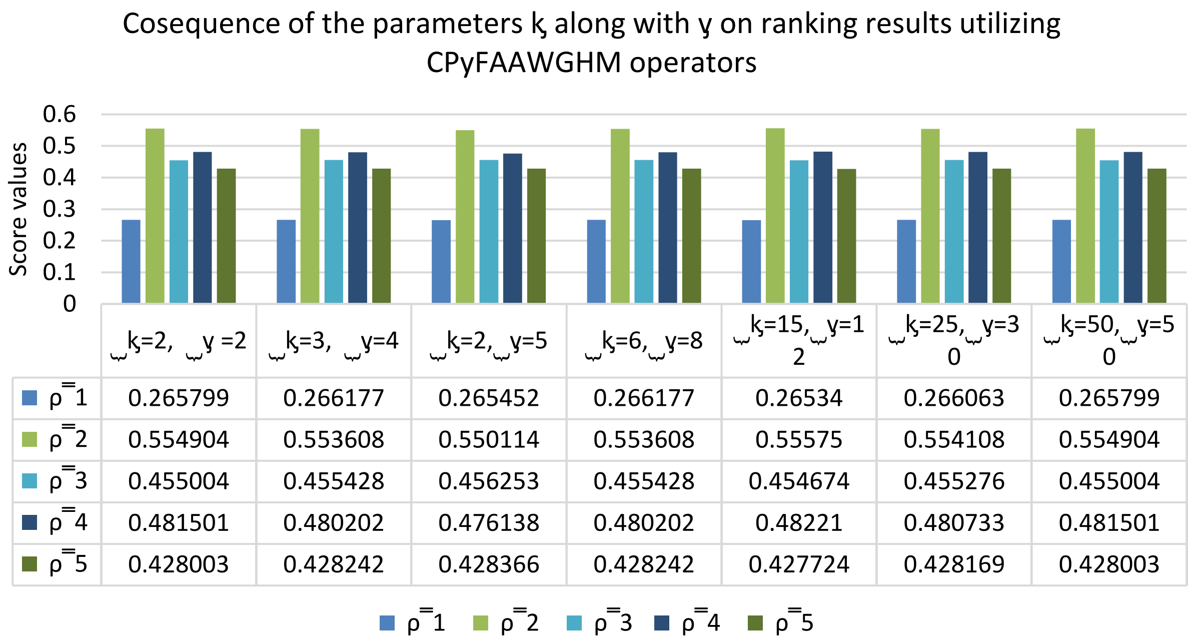


Fig. 6. Graph of the consequence of the parameters ķ and ỵ on ranking results of CPyFAAWGHM

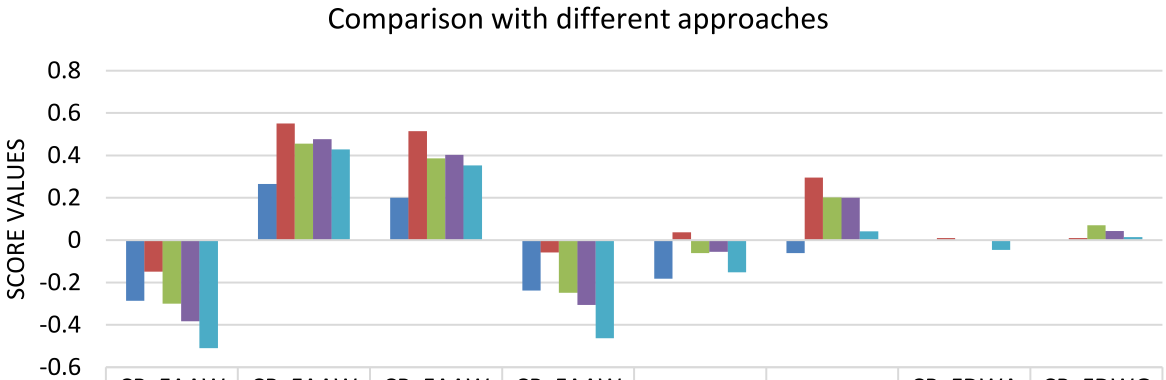
7. Comparative Analysis

Several researchers have used the concept of the MCDM. In this section, we perform a comparative analysis between the proposed AOs and several existing ones. The selected benchmarks for comparison are included the AOs of CPyFAAWA and CPyFAAWG by Hussain *et al.,* [62], AOs of CPyFWA and CPyFWG operators by Mehmood *et al.,* [23], and the AOs CPyFDWAA and CPyFDWOs by Akram *et al.,* [59]. The results of previously mentioned operators are displayed in Table 7 and Figure 7.

Table 7
The contrast between the discussed and current AOs

Aggregation operators	Score values					Ranking and ordering
	$\bar{\bar{C}}(\overline{\rho_1})$	$\bar{\bar{C}}(\overline{\rho_2})$	$\bar{\bar{C}}(\overline{\rho_3})$	$\bar{\bar{C}}(\overline{\rho_4})$	$\bar{\bar{C}}(\overline{\rho_5})$	
CPyFAAWHM	-0.2856	-0.1482	-0.2995	-0.3834	-0.51	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_1 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_5$
CPyFAAWGHM	0.26545	0.55011	0.45625	0.47614	0.42837	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_1$
CPyFAAWA[62]	0.20039	0.513937	0.386647	0.403329	0.353522	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_1$
CPyFAAWG[62]	-0.23837	-0.05805	-0.24874	-0.30559	-0.46381	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_1 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_5$
CPyFWA [23]	-0.18202	0.036798	-0.06043	-0.05531	-0.15129	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_1$
CPyFWG[23]	-0.06096	0.294807	0.201997	0.199732	0.041995	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_1$
CPyFDWAA[59]	-0.00175	0.010445	-0.00132	-0.00261	-0.04589	$\bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_1 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_5$
CPyFDWGA[59]	0.000325	0.009244	0.069778	0.043858	0.014016	$\bar{\bar{\rho}}_3 > \bar{\bar{\rho}}_4 > \bar{\bar{\rho}}_5 > \bar{\bar{\rho}}_2 > \bar{\bar{\rho}}_1$

The suggested aggregation operators based on the aggregation result in more balanced and stronger rankings within the framework of CPyFSs, where the uncertainty is very complex. It can be observed that the proposed operators provide more consistent and discriminative score values, enabling a clearer distinction among the alternatives. In particular, both CPyFAAWHM and CPyFAAWGHM yield stable and logically coherent ranking orders, with the alternative $\bar{\bar{\rho}}_2$ consistently identified as the best option across most cases, which aligns with the majority of existing methods and validates the reliability of the proposed framework. Moreover, unlike some existing operators (e.g., CPyFDWGA) that produce inconsistent or contradictory rankings, the proposed methods maintain robustness and sensitivity in handling complex Pythagorean fuzzy information. This improved performance is attributed to the incorporation of Heronian mean-based structures, which effectively capture the interrelationships among input arguments. Therefore, the proposed aggregation operators not only generalize existing models but also enhance decision-making accuracy and stability, highlighting their superiority and practical applicability in MCDM problems.



	CPyFAAWHM	CPyFAAWGHM	CPyFAAWA	CPyFAAWG	CPyFWA	CPyFWG	CPyFDWA	CPyFDWGA
$\bar{\bar{\rho}}_1$	-0.2856	0.265452	0.20039	-0.23837	-0.18202	-0.06096	-0.00175	0.000325
$\bar{\bar{\rho}}_2$	-0.14818	0.550114	0.513937	-0.05805	0.036798	0.294807	0.010445	0.009244
$\bar{\bar{\rho}}_3$	-0.29947	0.456253	0.386647	-0.24874	-0.06043	0.2019973	-0.00132	0.069778
$\bar{\bar{\rho}}_4$	-0.3834	0.476138	0.403329	-0.30559	-0.05531	0.199732	-0.00261	0.043858
$\bar{\bar{\rho}}_5$	-0.50999	0.428366	0.353522	-0.46381	-0.15129	0.041995	-0.045885	0.014016

$\bar{\bar{\rho}}_1$ $\bar{\bar{\rho}}_2$ $\bar{\bar{\rho}}_3$ $\bar{\bar{\rho}}_4$ $\bar{\bar{\rho}}_5$

Fig. 7. Contrast with different approaches

8. Conclusion

In this study, we developed a family of novel aggregation operators under the Complex Pythagorean Fuzzy Sets framework, namely CPyFAAHM, CPyFAAGHM, CPyFAAWHM, and CPyFAAWGHM. These operators integrate Aczel–Alsina operational laws with Heronian mean structures to effectively capture interrelationships among input arguments. The proposed operators were mathematically validated by proving key properties such as idempotency, boundedness, and monotonicity, ensuring their theoretical consistency. MCDM is widely applied in various fields, including social sciences, economics, and medical science, to address complex decision-making challenges. Thus, a practical MCDM model was constructed and applied to a decision-making problem, where the alternative $\bar{\bar{\rho}}_2$ was consistently identified as the optimal choice. Comparative analysis demonstrated that the proposed operators produce more stable, consistent, and discriminative ranking results compared to existing methods. Furthermore, sensitivity analysis revealed that although parameter variations affect score values, the final ranking remains largely stable, confirming the robustness and reliability of the proposed approach.

8.1 Novelty of the Study

The novelty of this study lies in the development of a new family of Complex Pythagorean Fuzzy (CPyF) aggregation operators, namely CPyFAAHM, CPyFAAGHM, CPyFAAWHM, and CPyFAAWGHM, which significantly enhance decision-making under uncertainty. A key innovative aspect is the firsttime integration of Aczel–Alsina operational laws with Heronian mean and geometric Heronian mean structures within the CPyF framework, enabling a more flexible and generalized aggregation process. Unlike conventional operators, the proposed models effectively capture the interrelationships among attributes while simultaneously handling complex-valued information that incorporates both amplitude and phase components. This dual capability allows for a more realistic representation of uncertain and periodic data. Furthermore, the proposed operators demonstrate improved performance by producing more stable and consistent ranking results, along with better discrimination among competing alternatives. Sensitivity analysis also confirms the robustness of the approach, as the ranking of alternatives remains largely unaffected under variations in parameter values, highlighting the reliability and practical applicability of the proposed methodology in complex multi-criteria decision-making problems.

8.2 Future Research Directions and Limitations

The Complex Pythagorean fuzzy sets is an efficient tool to handle two-dimensional periodic uncertain information, which has various applications in the fields of fuzzy modeling and decision making. The developed framework demonstrates notable improvements over conventional fuzzy models, specifically by resolving key limitations associated with FSs, IFSs, PyFSs, CFSs, and CIFSs. Although the proposed approach demonstrates strong effectiveness, it is not without limitations, which open avenues for further improvements. The primary limitations are summarized below:

i. The constraint on the square sum of complex value membership and complex value non-membership function (i.e., one) is the limitation of the proposed model, the decision-makers have to choose the values so that the square sum of complex value membership and complex value non-membership function is less than or equal to one.
ii. The idea of CPyFSs is a modified version of the numerous existing techniques, such as PyFSs and CIFSs. Still, they also have many limitations, for instance, when an expert provides information in the form of yes, no, abstinence, and refusal. The existing technique of CPyFSs has failed because of limited features. The technique of complex pythagorean fuzzy information has just yes and no but not others, and due to this reason, they failed to cope with many problems.
iii. Moreover, the proposed model assumes precise selection of parameters, such as weighting factors and operational parameters, which may significantly influence the final decision results and introduce subjectivity.
iv. In addition, the computational complexity of the approach is relatively high due to the involvement of complex-valued data (amplitude and phase components) and the use of hybrid aggregation structures, which may limit its applicability in large-scale or real-time decision-making problems.

In the future, we will show the applicability of the suggested AOs for CPyFSs in pattern recognition and clustering analysis. We will also modify our principles in this analysis based on CPyFSs for Complex Picture Fuzzy Sets, Complex Spherical Fuzzy Sets, and complex T-Spherical Fuzzy Sets. To address distinctive aspects of decision-making problems, it is essential to promote further research in fuzzy set theory [62] as well as bipolar fuzzy set frameworks [63, 64]. In addition, future work will focus on applying recently developed algorithms to more sophisticated and complex decision-making environments [65–69].

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Conflict of Interest

There is no conflict of interest to disclose.

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