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Hamacher Aggregation Operators for Pythagorean Fuzzy Set and its Application in Multi-Attribute Decision-Making Problem

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ABSTRACT

Pythagorean fuzzy set is a useful expansion of intuitionistic fuzzy set for dealing with ambiguities, which mostly occur in real-life problems. Hamacher t-norm also has important and compatible norms that incorporate a parameter that offers various options to decision-makers during the information fusion process, thereby enhancing their ability to model decision-making problems effectively compared to alternative methods. In this study, Hamacher operators are being used to introduce several Pythagorean fuzzy Hamacher interactive weighted averaging (PFHIWA), Pythagorean fuzzy Hamacher interactive ordered weighted averaging (PFHIOWA), Pythagorean fuzzy Hamacher interactive weighted geometric (PFHIWG), and Pythagorean fuzzy Hamacher interactive ordered weighted geometric (PFHIOWG) operators. The properties of these operators are examined in detail. The benefit of using progressive operators is that they deliver more understanding of the scenario to the decision-makers. Proposed operators are utilized to elaborate multi-attribute decision-making (MADM). By showing the sensitivity analysis, our proposed operator has high stability related to multi-attribute decision-making (MADM) under the Pythagorean fuzzy data set.

1. Introduction

Decision-making (DM) is vital for problem-solving, achieving goals, and allocating resources effectively. It facilitates innovation, risk management, and conflict resolution while empowering individuals and organizations to adapt to change and drive continuous improvement. Zadeh [1] was the first one to introduce the concept of fuzzy set (FS), which only had a membership degree (MDg). It got more attention as FS theory deals with ambiguity in the data. Then, Atanassov [2] extended this concept of FS and gave the idea of intuitionistic FS (IFS) in which there was a non-membership degree (NMDg) alongside MDg. This boosts the study of FS, and many researchers have started paying attention to this area. However, IFS had some restrictions, such as the sum of MDg and NMDg

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being between 0 and 1. To overcome this issue, Yager [3] presented the idea of a Pythagorean fuzzy set (PyFS).

In PyFS, the sum of the square of MDg and NMDg is between 0 and 1, i.e., $0 \leq (MDg)^2 + (NMDg)^2 \leq 1$ (Figure 1). Yager [4] defined the operational rules and other properties of PyFS and compared them with IFS. Peng and Yang [5] proposed division and subtraction for PyFS and developed a superiority and inferiority ranking method for PyFS. Yager and Abbasov [6] developed the averaging and geometric aggregation operators (AOs) based on PyFS. In their study, Zhang and Xu [7] used the TOPSIS method on PyFS to solve the DM problems. Bibliometric analysis of PyFS was done by Lin *et al.,* [8]. Li and Lu [9] presented some unique similarities and distance measures of PyFS and utilized them in issues related to real life. Circular PyFS was designed by Olgun and Ünver [10] and showed their applications by using these operators in multi-attribute DM (MADM) issues. Akram *et al.,* [11] introduced a new optimization method under the PyFS environment. In the same, many researchers have done work based on PyFS [12-17].

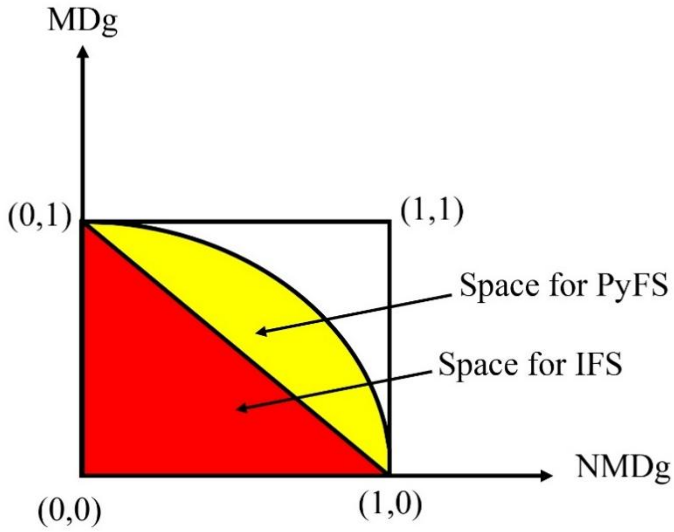

Fig. 1. Pythagorean fuzzy set

From those mentioned above, it is noticed that different researchers used different approaches to strengthen their studies. Overall, the customary AOs usually assumed simple algebraic rules. T-norms (t-Nr) and t-conorms (t-CNr) are widely used in FS and are important because of algebraic expressions. Einstein Nr [18], Archimedean Nr [19], and Frank Nr [20] have been widely used in developing different AOs in FS. Sarfraz [21] introduces interval-valued T-spherical fuzzy Dombi Hamy mean operators as a powerful tool for group decision-making by using Dombi Nr. Tešić and Marinković [22] defined normalized aggregation operators using Bonferroni Mean operators. Hamacher t-Nr [23] also has important and compatible Nr. These norms incorporate a parameter that offers various options to decision-makers during the information fusion process, thereby enhancing their ability to model decision-making problems effectively compared to alternative methods. Huang [24] developed AOs under the Hamacher t-Nr environment. Tang *et al.,* [25] studied the analysis of Hamacher AOs and their effect on MADM problems. Linguistic q-ROFS AOs based on Hamacher t-Nr were introduced by Deb *et al.,* [26]. Garg [27] also used Hamcher T-Nr to develop some new AOs and used these AOs by entropy method for the MADM issue. There are useful studies [28-32] to understand Hamacher t-Nr.

Building upon recognized research deficiencies, we propose AOs tailored for PyFS. Our study unveils weighted averaging and geometric AOs, leveraging Hamacher norm operations. These operators cater to preferences articulated through PyFS. Notably, the relative ranking of alternatives hinges significantly on the choice of weight vectors allocated to PyFSs, underscoring the importance of discerning suitable weight assignments. This contribution seeks to address gaps in DM methodologies by offering refined tools for handling uncertainty and ambiguity.

The pattern of the paper is as follows: in Section 2, there are some fundamental concepts and definitions. Section 3 presents improved operations on Hamacher t-Nr under the PyFS environment and introduces PFH interactive weighted averaging (PFHIWA), PFH interactive ordered weighted averaging (PFHIOWA), PFH interactive weighted geometric (PFHIWG) and PFH interactive ordered weighted geometric (PFHIOWG) operators respectively. The MADM technique has been briefly discussed in Section 4 to tackle our proposed operators. In section 5, we have shown the applications of our proposed operators by a practical example. Sections 6 and 7 are devoted to sensitivity analysis and comparative analysis to show the stability and importance of our operators. In Section 8, the conclusion of this work provides insights into important findings and suggests prospective directions for further research.

2. Preliminaries

This section will discuss some basic definitions and rules that will help construct our paper.

Definition 1 [2]. An IFS T defined in Y is an ordered pain presented by

$$T = \{\langle y, g(y), \mathrm{h}(y)\rangle | y \in Y\}$$

where $g(y), \mathrm{h}(y): Y \to [0,1]$ represent the MDg and NMDg of the element y such that $g(y), \mathrm{h}(y) \in [0,1]$ and $g(y) + \mathrm{h}(y) \leq 1$ for all y.

Definition 2 [3]. A PyFS T defined in Y is an ordered pain presented by

$$T = \{\langle y, g(y), \mathrm{h}(y)\rangle | y \in Y\}$$

where $g(y), \mathrm{h}(y): Y \to [0,1]$ represent the MDg and NMDg of the element y such that $g(y), \mathrm{h}(y) \in [0,1]$ and $(g(y))^2 + (\mathrm{h}(y))^2 \leq 1$ for all y.

Definition 3 [5]. The score function SF for any PyFN is given by

$$SF = (g(y))^2 - \left(\mathrm{h}(y)\right)^2 \quad (1)$$

when SF of two PyFNs, the accuracy function (AF) is used to compare them.

Definition 4. The AF for any PyFN is given by

$$AF = (g(y))^2 + \left(\mathrm{h}(y)\right)^2 \quad (2)$$

Definition 5 [19]. A function $G: [0,1] \times [0,1] \to [0,1]$ is known as t-Nr if it holds the properties of monotonicity, commutativity, boundary conditions, and associativity. On the contrary, a function Q defined by $Q(o,p) = 1 - G(1-o, 1-p)$ for all $o, p \in [0,1]$ is known as t-CNr.

Definition 6 [19]. Archimedean t-Nr function G is a continuous t-Nr which satisfies the condition G(o,o) > o for o ∈ (0,1). A strict Archimedean t-Nr is strictly increasing t-Nr.

Definition 7 [23]. Garg defined the bnorm operations by considering $w(t) = ln\left(\frac{\mathfrak{Y}+(1-\mathfrak{Y})t}{t}\right), \mathfrak{Y} \in (0,\infty)$ for two different IFN $\underline{\omega}_1 = \langle g_1, \mathrm{h}_1\rangle$ and $\underline{\omega}_2 = \langle g_2, \mathrm{h}_2\rangle$ such as:

(i) $\underline{\omega}_1 \oplus \underline{\omega}_2 = \left(\frac{g_1+g_2-g_1g_2-(1-\mathfrak{Y})g_1g_2}{1-(1-\mathfrak{Y})g_1g_2}, \frac{\mathrm{h}_1\mathrm{h}_2}{\mathfrak{Y}+(1-\mathfrak{Y})(\mathrm{h}_1+\mathrm{h}_2-\mathrm{h}_1\mathrm{h}_2)}\right)$

(ii) $\underline{\omega}_1 \otimes \underline{\omega}_2 = \left(\frac{g_1g_2}{\mathfrak{Y}+(1-\mathfrak{Y})(g_1+g_2-g_1g_2)}, \frac{\mathrm{h}_1+\mathrm{h}_2-\mathrm{h}_1\mathrm{h}_2-(1-\mathfrak{Y})\mathrm{h}_1\mathrm{h}_2}{1-(1-\mathfrak{Y})\mathrm{h}_1\mathrm{h}_2}\right)$

Definition 8 [24]. For a set of IFNS $\underline{\omega}_\mathfrak{i} = \langle g_\mathfrak{i}, \mathrm{h}_\mathfrak{i}\rangle (\mathfrak{i} = 1,2,\ldots,\dot{n})$, Huang proposed AOs, which are defined as below

$$IFHWA(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) =$$

$$\left(\frac{\bigoplus_{\imath=1}^{\dot{n}} [1+(\mathfrak{Y}-1)g_\imath]^{đ_\imath} - \bigoplus_{\imath=1}^{\dot{n}} (1-g_\imath)^{đ_\imath}}{\bigoplus_{\imath=1}^{\dot{n}} [1+(\mathfrak{Y}-1)g_\imath]^{đ_\imath} - (\mathfrak{Y}-1)\bigoplus_{\imath=1}^{\dot{n}} (1-g_\imath)^{đ_\imath}}, \frac{\mathfrak{Y} \bigoplus_{\imath=1}^{\dot{n}} h_\imath^{đ_\imath}}{\bigoplus_{\imath=1}^{\dot{n}} [1+(\mathfrak{Y}-1)(1-h_\imath)]^{đ_\imath} - (\mathfrak{Y}-1)\bigoplus_{\imath=1}^{\dot{n}} h_\imath^{đ_\imath}}\right)$$

and

$$IFHWg(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) =$$

$$\left(\frac{\mathfrak{Y} \bigoplus_{\imath=1}^{\dot{n}} g_\imath^{đ_\imath}}{\bigoplus_{\imath=1}^{\dot{n}} [1+(\mathfrak{Y}-1)(1-g_\imath)]^{đ_\imath} - (\mathfrak{Y}-1)\bigoplus_{\imath=1}^{\dot{n}} g_\imath^{đ_\imath}}, \frac{\bigoplus_{\imath=1}^{\dot{n}} [1+(\mathfrak{Y}-1)h_\imath]^{đ_\imath} - \bigoplus_{\imath=1}^{\dot{n}} (1-h_\imath)^{đ_\imath}}{\bigoplus_{\imath=1}^{\dot{n}} [1+(\mathfrak{Y}-1)h_\imath]^{đ_\imath} - (\mathfrak{Y}-1)\bigoplus_{\imath=1}^{\dot{n}} (1-h_\imath)^{đ_\imath}}\right).$$

Definition 9 [27]. For a set of IFNS $\underline{\omega}_\imath = \langle g_\imath, h_\imath \rangle (\imath = 1,2,\dots,\dot{n})$, Garg proposed improved AOs, which was shown in [24], which are defined as

$$IFHIWA(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) =$$

$$\left(\frac{\bigoplus_{\imath=1}^{\dot{n}} [1+(\mathfrak{Y}-1)g_\imath]^{đ_\imath} - \bigoplus_{\imath=1}^{\dot{n}} (1-g_\imath)^{đ_\imath}}{\bigoplus_{\imath=1}^{\dot{n}} [1+(\mathfrak{Y}-1)g_\imath]^{đ_\imath} - (\mathfrak{Y}-1)\bigoplus_{\imath=1}^{\dot{n}} (1-g_\imath)^{đ_\imath}}, \frac{\mathfrak{Y} \bigoplus_{\imath=1}^{\dot{n}} (1-g_\imath)^{đ_\imath} - \mathfrak{Y} \bigoplus_{\imath=1}^{\dot{n}} (1-g_\imath-h_\imath)^{đ_\imath}}{\bigoplus_{\imath=1}^{\dot{n}} [1+(\mathfrak{Y}-1)g_\imath]^{đ_\imath} - (\mathfrak{Y}-1)\bigoplus_{\imath=1}^{\dot{n}} (1-g_\imath)^{đ_\imath}}\right)$$

and

$$IFHIWG(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) =$$

$$\left(\frac{\mathfrak{Y} \bigoplus_{\imath=1}^{\dot{n}} (1-h_\imath)^{đ_\imath} - \mathfrak{Y} \bigoplus_{\imath=1}^{\dot{n}} (1-g_\imath-h_\imath)^{đ_\imath}}{\bigoplus_{\imath=1}^{\dot{n}} [1+(\mathfrak{Y}-1)h_\imath]^{đ_\imath} - (\mathfrak{Y}-1)\bigoplus_{\imath=1}^{\dot{n}} (1-h_\imath)^{đ_\imath}}, \frac{\bigoplus_{\imath=1}^{\dot{n}} [1+(\mathfrak{Y}-1)h_\imath]^{đ_\imath} - \bigoplus_{\imath=1}^{\dot{n}} (1-h_\imath)^{đ_\imath}}{\bigoplus_{\imath=1}^{\dot{n}} [1+(\mathfrak{Y}-1)h_\imath]^{đ_\imath} - (\mathfrak{Y}-1)\bigoplus_{\imath=1}^{\dot{n}} (1-h_\imath)^{đ_\imath}}\right).$$

3. Improved Operational Laws and Aggregation Operators for Pythagorean Fuzzy Set

We will present some new rules and laws which will be helpful in the study of AOs.

Definition 10. Suppose $\underline{\omega} = \langle g, h \rangle$ and $\underline{\omega}_\imath = \langle g_\imath, h_\imath \rangle (\imath = 1,2,\dots,\dot{n})$ be a set of PyFNs, then the new operational laws using the Hamacher norm for these numbers can be defined as:

(i) $\underline{\omega}_1 \oplus \underline{\omega}_2 \dots \oplus \underline{\omega}_\imath =$
$$\left(\sqrt{\frac{\bigoplus_{\imath=1}^{\dot{n}}[1+(\mathfrak{Y}-1)(g_\imath)^2] - \bigoplus_{\imath=1}^{\dot{n}}(1-(g_\imath)^2)}{\bigoplus_{\imath=1}^{\dot{n}}[1+(\mathfrak{Y}-1)(g_\imath)^2] - (\mathfrak{Y}-1)\bigoplus_{\imath=1}^{\dot{n}}(1-(g_\imath)^2)}}, \sqrt{\frac{\mathfrak{Y}\bigoplus_{\imath=1}^{\dot{n}}(1-(g_\imath)^2) - \mathfrak{Y}\bigoplus_{\imath=1}^{\dot{n}}(1-(g_\imath)^2-(h_\imath)^2)}{\bigoplus_{\imath=1}^{\dot{n}}[1+(\mathfrak{Y}-1)(g_\imath)^2] - (\mathfrak{Y}-1)\bigoplus_{\imath=1}^{\dot{n}}(1-(g_\imath)^2)}}\right);$$

(ii) $\underline{\omega}_1 \otimes \underline{\omega}_2 \dots \otimes \underline{\omega}_\imath =$
$$\left(\sqrt{\frac{\mathfrak{Y}\bigoplus_{\imath=1}^{\dot{n}}(1-(h_\imath)^2) - \mathfrak{Y}\bigoplus_{\imath=1}^{\dot{n}}(1-(g_\imath)^2-(h_\imath)^2)}{\bigoplus_{\imath=1}^{\dot{n}}[1+(\mathfrak{Y}-1)(h_\imath)^2] - (\mathfrak{Y}-1)\bigoplus_{\imath=1}^{\dot{n}}(1-(h_\imath)^2)}}, \sqrt{\frac{\bigoplus_{\imath=1}^{\dot{n}}[1+(\mathfrak{Y}-1)(h_\imath)^2] - \bigoplus_{\imath=1}^{\dot{n}}(1-(h_\imath)^2)}{\bigoplus_{\imath=1}^{\dot{n}}[1+(\mathfrak{Y}-1)(h_\imath)^2] - (\mathfrak{Y}-1)\bigoplus_{\imath=1}^{\dot{n}}(1-(h_\imath)^2)}}\right);$$

(iii) $$þ\underline{\omega} = \left(\sqrt{\frac{[1+(\mathfrak{Y}-1)(g)^2]^þ - (1-(g)^2)^þ}{[1+(\mathfrak{Y}-1)(g)^2]^þ - (\mathfrak{Y}-1)(1-(g)^2)^þ}}, \sqrt{\frac{\mathfrak{Y}(1-(g)^2)^þ - \mathfrak{Y}(1-(g)^2-(h)^2)^þ}{[1+(\mathfrak{Y}-1)(g)^2]^þ - (\mathfrak{Y}-1)(1-(g)^2)^þ}}\right);$$

(iv) $$\underline{\omega}^þ = \left(\sqrt{\frac{\mathfrak{Y}(1-(h)^2)^þ - \mathfrak{Y}(1-(g)^2-(h)^2)^þ}{[1+(\mathfrak{Y}-1)(h)^2]^þ - (\mathfrak{Y}-1)(1-(h)^2)^þ}}, \sqrt{\frac{[1+(\mathfrak{Y}-1)(h)^2]^þ - (1-(h)^2)^þ}{[1+(\mathfrak{Y}-1)(h)^2]^þ - (\mathfrak{Y}-1)(1-(h)^2)^þ}}\right).$$

where $þ > 0$ and $\mathfrak{Y} \in (0, \infty)$ are the real numbers.

It can be seen in Definition 10 that the sum operation has become more optimistic than the sum defined in *Definition 7*. Hence, the behavior is more inclined toward the MD than the NMD. So, the result will be more accurate.

3.1 Averaging Operators

For the set of $\underline{\omega}_\imath = \langle \mathscr{g}_\imath, \mathrm{h}_\imath \rangle (\imath = 1,2, \dots, \dot{n})$ with $\mathscr{g}_\imath \neq 1$ for all $\imath$ and relying on the above-mentioned rules, we will present new PFH interactive weighted (PFHIWA) and order weighted (PFHIOWA) averaging operators.

Definition 11. Suppose Y be the collection of PyNFs and $PFHIWA: \Upsilon^{\dot{n}} \rightarrow \Upsilon$, such that:

$$PFHIWA(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) = \text{đ}_1 \underline{\omega}_1 \oplus \text{đ}_2 \underline{\omega}_2 \oplus \dots \oplus \text{đ}_{\dot{n}} \underline{\omega}_{\dot{n}} \qquad (3)$$

where $\text{đ} = (\text{đ}_1, \text{đ}_2, \dots, \text{đ}_{\dot{n}})^T$ is the weight vector of $\underline{\omega}_\imath$ such that $\text{đ}_\imath > 0$ and $\sum_{\imath=1}^{\dot{n}} \text{đ}_\imath = 1$; then, PFHIWA is defined as the PFH interactive weighted averaging operator.

Theorem 1. For PyFNs $\underline{\omega}_\imath = \langle \mathscr{g}_\imath, \mathrm{h}_\imath \rangle$ with $\mathscr{g}_\imath \neq 1 (\imath = 1,2, \dots, \dot{n})$, the aggregated value by using PFHIWA is again a PyFN and is defined as:

$$PFHIWA(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) =$$

$$\left(\sqrt{\frac{\oplus_{\imath=1}^{\dot{n}} [1+(\mathfrak{Y}-1)(\mathscr{g}_\imath)^2]^{\text{đ}_\imath} - \oplus_{\imath=1}^{\dot{n}} (1-(\mathscr{g}_\imath)^2)^{\text{đ}_\imath}}{\oplus_{\imath=1}^{\dot{n}} [1+(\mathfrak{Y}-1)(\mathscr{g}_\imath)^2]^{\text{đ}_\imath} - (\mathfrak{Y}-1) \oplus_{\imath=1}^{\dot{n}} (1-(\mathscr{g}_\imath)^2)^{\text{đ}_\imath}}}, \sqrt{\frac{\mathfrak{Y} \oplus_{\imath=1}^{\dot{n}} (1-(\mathscr{g}_\imath)^2)^{\text{đ}_\imath} - \mathfrak{Y} \oplus_{\imath=1}^{\dot{n}} (1-(\mathscr{g}_\imath)^2-(\mathrm{h}_\imath)^2)^{\text{đ}_\imath}}{\oplus_{\imath=1}^{\dot{n}} [1+(\mathfrak{Y}-1)(\mathscr{g}_\imath)^2]^{\text{đ}_\imath} - (\mathfrak{Y}-1) \oplus_{\imath=1}^{\dot{n}} (1-(\mathscr{g}_\imath)^2)^{\text{đ}_\imath}}} \right) \qquad (4)$$

Proof: We will prove Theorem 1 by mathematical induction. So, we have

I. When $\dot{n} = 2$, we have $\mathscr{g}_1 + \mathscr{g}_2 = 1$ and $PFHIWA(\underline{\omega}_1, \underline{\omega}_2) = \text{đ}_1 \underline{\omega}_1 \oplus \text{đ}_2 \underline{\omega}_2$. Now, for PyFNs $\underline{\omega}_1$ and $\underline{\omega}_2$, we have

$$\text{đ}_1 \underline{\omega}_1 = \left(\sqrt{\frac{\left[1+(\mathfrak{Y}-1)(\mathscr{g}_1)^2\right]^{\text{đ}_1} - \left(1-(\mathscr{g}_1)^2\right)^{\text{đ}_1}}{\left[1+(\mathfrak{Y}-1)(\mathscr{g}_1)^2\right]^{\text{đ}_1} - (\mathfrak{Y}-1)\left(1-(\mathscr{g}_1)^2\right)^{\text{đ}_1}}}, \sqrt{\frac{\mathfrak{Y}\left(1-(\mathscr{g}_1)^2\right)^{\text{đ}_1} - \mathfrak{Y}\left(1-(\mathscr{g}_1)^2-(\mathrm{h}_1)^2\right)^{\text{đ}_1}}{\left[1+(\mathfrak{Y}-1)(\mathscr{g}_1)^2\right]^{\text{đ}_1} - (\mathfrak{Y}-1)\left(1-(\mathscr{g}_1)^2\right)^{\text{đ}_1}}} \right)$$

and

$$\text{đ}_2 \underline{\omega}_2 = \left(\sqrt{\frac{\left[1+(\mathfrak{Y}-1)(\mathscr{g}_2)^2\right]^{\text{đ}_2} - \left(1-(\mathscr{g}_2)^2\right)^{\text{đ}_2}}{\left[1+(\mathfrak{Y}-1)(\mathscr{g}_2)^2\right]^{\text{đ}_2} - (\mathfrak{Y}-1)\left(1-(\mathscr{g}_2)^2\right)^{\text{đ}_2}}}, \sqrt{\frac{\mathfrak{Y}\left(1-(\mathscr{g}_2)^2\right)^{\text{đ}_2} - \mathfrak{Y}\left(1-(\mathscr{g}_2)^2-(\mathrm{h}_2)^2\right)^{\text{đ}_2}}{\left[1+(\mathfrak{Y}-1)(\mathscr{g}_2)^2\right]^{\text{đ}_2} - (\mathfrak{Y}-1)\left(1-(\mathscr{g}_2)^2\right)^{\text{đ}_2}}} \right)$$

So, we have

$$PFHIWA(\underline{\omega}_1, \underline{\omega}_2) = \text{đ}_1 \underline{\omega}_1 \oplus \text{đ}_2 \underline{\omega}_2$$

$$= \left(\sqrt{\frac{[1+(\mathfrak{Y}-1)(\mathscr{g}_1)^2]^{\text{đ}_1} - (1-(\mathscr{g}_1)^2)^{\text{đ}_1}}{[1+(\mathfrak{Y}-1)(\mathscr{g}_1)^2]^{\text{đ}_1} - (\mathfrak{Y}-1)(1-(\mathscr{g}_1)^2)^{\text{đ}_1}}}, \sqrt{\frac{\mathfrak{Y}(1-(\mathscr{g}_1)^2)^{\text{đ}_1} - \mathfrak{Y}(1-(\mathscr{g}_1)^2-(\mathrm{h}_1)^2)^{\text{đ}_1}}{[1+(\mathfrak{Y}-1)(\mathscr{g}_1)^2]^{\text{đ}_1} - (\mathfrak{Y}-1)(1-(\mathscr{g}_1)^2)^{\text{đ}_1}}} \right)$$

$$\oplus \left(\sqrt{\frac{[1+(\mathfrak{Y}-1)(\mathscr{g}_2)^2]^{\text{đ}_2} - (1-(\mathscr{g}_2)^2)^{\text{đ}_2}}{[1+(\mathfrak{Y}-1)(\mathscr{g}_2)^2]^{\text{đ}_2} - (\mathfrak{Y}-1)(1-(\mathscr{g}_2)^2)^{\text{đ}_2}}}, \sqrt{\frac{\mathfrak{Y}(1-(\mathscr{g}_2)^2)^{\text{đ}_2} - \mathfrak{Y}(1-(\mathscr{g}_2)^2-(\mathrm{h}_2)^2)^{\text{đ}_2}}{[1+(\mathfrak{Y}-1)(\mathscr{g}_2)^2]^{\text{đ}_2} - (\mathfrak{Y}-1)(1-(\mathscr{g}_2)^2)^{\text{đ}_2}}} \right)$$

$$= \left(\sqrt{\frac{\oplus_{\imath=1}^{2} [1+(\mathfrak{Y}-1)(\mathscr{g}_\imath)^2]^{\text{đ}_\imath} - \oplus_{\imath=1}^{2} (1-(\mathscr{g}_\imath)^2)^{\text{đ}_\imath}}{\oplus_{\imath=1}^{2} [1+(\mathfrak{Y}-1)(\mathscr{g}_\imath)^2]^{\text{đ}_\imath} - (\mathfrak{Y}-1) \oplus_{\imath=1}^{2} (1-\mathscr{g}_\imath)^{\text{đ}_\imath}}}, \sqrt{\frac{\mathfrak{Y} \oplus_{\imath=1}^{2} (1-(\mathscr{g}_\imath)^2)^{\text{đ}_\imath} - \mathfrak{Y} \oplus_{\imath=1}^{2} (1-(\mathscr{g}_\imath)^2-(\mathrm{h}_\imath)^2)^{\text{đ}_\imath}}{\oplus_{\imath=1}^{2} [1+(\mathfrak{Y}-1)(\mathscr{g}_\imath)^2]^{\text{đ}_\imath} - (\mathfrak{Y}-1) \oplus_{\imath=1}^{2} (1-(\mathscr{g}_\imath)^2)^{\text{đ}_\imath}}} \right)$$

So, Eq. (4) is true for $\dot{n} = 2$

II. Suppose that Eq. (4) is also true for $\dot{n} = \kappa$

$$PFHIWA(\underline{\omega}_1, \underline{\omega}_2 \ldots, \underline{\omega}_{\dot{n}}) =$$

$$\left(\sqrt{\frac{\oplus_{i=1}^{\kappa}\left[1+(\mathfrak{Y}-1)\left(g_i\right)^2\right]^{đ_i} - \oplus_{i=1}^{\kappa}\left(1-\left(g_i\right)^2\right)^{đ_i}}{\oplus_{i=1}^{\kappa}\left[1+(\mathfrak{Y}-1)\left(g_i\right)^2\right]^{đ_i} - (\mathfrak{Y}-1)\oplus_{i=1}^{\kappa}\left(1-\left(g_i\right)^2\right)^{đ_i}}}, \sqrt{\frac{\mathfrak{Y}\oplus_{i=1}^{\kappa}\left(1-\left(g_i\right)^2\right)^{đ_i} - \mathfrak{Y}\oplus_{i=1}^{\kappa}\left(1-\left(g_i\right)^2-\left(h_i\right)^2\right)^{đ_i}}{\oplus_{i=1}^{\kappa}\left[1+(\mathfrak{Y}-1)\left(g_i\right)^2\right]^{đ_i} - (\mathfrak{Y}-1)\oplus_{i=1}^{\kappa}\left(1-\left(g_i\right)^2\right)^{đ_i}}}\right)$$

III. Now, for $\dot{n} = \kappa + 1$, we have

$$PFHIWA(\underline{\omega}_1, \underline{\omega}_2 \ldots, \underline{\omega}_{\kappa+1}) = \oplus_{i=1}^{\kappa+1} đ_i \underline{\omega}_i = PFHIWA(\underline{\omega}_1, \underline{\omega}_2 \ldots, \underline{\omega}_{\kappa}) \oplus đ_{\kappa+1}\underline{\omega}_{\kappa+1}$$

$$= \left(\sqrt{\frac{\oplus_{i=1}^{\kappa}[1+(\mathfrak{Y}-1)(g_i)^2]^{đ_i} - \oplus_{i=1}^{\kappa}(1-(g_i)^2)^{đ_i}}{\oplus_{i=1}^{\kappa}[1+(\mathfrak{Y}-1)(g_i)^2]^{đ_i} - (\mathfrak{Y}-1)\oplus_{i=1}^{\kappa}(1-(g_i)^2)^{đ_i}}}, \sqrt{\frac{\mathfrak{Y}\oplus_{i=1}^{\kappa}(1-(g_i)^2)^{đ_i} - \mathfrak{Y}\oplus_{i=1}^{\kappa}(1-(g_i)^2-(h_i)^2)^{đ_i}}{\oplus_{i=1}^{\kappa}[1+(\mathfrak{Y}-1)(g_i)^2]^{đ_i} - (\mathfrak{Y}-1)\oplus_{i=1}^{\kappa}(1-(g_i)^2)^{đ_i}}}\right)$$

$$\oplus \left(\sqrt{\frac{[1+(\mathfrak{Y}-1)(g_{\kappa+1})^2]^{đ_{\kappa+1}} - (1-(g_{\kappa+1})^2)^{đ_{\kappa+1}}}{[1+(\mathfrak{Y}-1)(g_{\kappa+1})^2]^{đ_{\kappa+1}} - (\mathfrak{Y}-1)(1-(g_{\kappa+1})^2)^{đ_{\kappa+1}}}}, \sqrt{\frac{\mathfrak{Y}(1-(g_{\kappa+1})^2)^{đ_{\kappa+1}} - \mathfrak{Y}(1-(g_{\kappa+1})^2-(h_{\kappa+1})^2)^{đ_{\kappa+1}}}{[1+(\mathfrak{Y}-1)(g_{\kappa+1})^2]^{đ_{\kappa+1}} - (\mathfrak{Y}-1)(1-(g_{\kappa+1})^2)^{đ_{\kappa+1}}}}\right)$$

$$= \left(\sqrt{\frac{\oplus_{i=1}^{\kappa+1}[1+(\mathfrak{Y}-1)(g_i)^2]^{đ_i} - \oplus_{i=1}^{\kappa+1}(1-(g_i)^2)^{đ_i}}{\oplus_{i=1}^{\kappa+1}[1+(\mathfrak{Y}-1)(g_i)^2]^{đ_i} - (\mathfrak{Y}-1)\oplus_{i=1}^{\kappa+1}(1-(g_i)^2)^{đ_i}}}, \sqrt{\frac{\mathfrak{Y}\oplus_{i=1}^{\kappa+1}(1-(g_i)^2)^{đ_i} - \mathfrak{Y}\oplus_{i=1}^{\kappa+1}(1-(g_i)^2-(h_i)^2)^{đ_i}}{\oplus_{i=1}^{\kappa+1}[1+(\mathfrak{Y}-1)(g_i)^2]^{đ_i} - (\mathfrak{Y}-1)\oplus_{i=1}^{\kappa+1}(1-(g_i)^2)^{đ_i}}}\right)$$

So, Eq. (4) also holds for $\dot{n} = \kappa + 1$.

Hence, it is proved that the Eq.(4) holds for all positive integer $\dot{n}$.

Now, we will define some properties for PFHIWA.

Theorem 2 (Idempotency). If $\underline{\omega}_i = \underline{\omega}_0$ for all i then $PFHIWA(\underline{\omega}_1, \underline{\omega}_2 \ldots, \underline{\omega}_{\dot{n}}) = \underline{\omega}_0$.

Proof: For a set of PyFNs $\underline{\omega}_i(i = 1,2, \ldots, \dot{n})$ and weight vector $đ_i$ such that $đ_i > 0$ and $\sum_{i=1}^{\dot{n}} đ_i = 1$. Suppose $\underline{\omega}_i = \underline{\omega}_0$ for all i then by PFHIWA operator, we have

$$\begin{aligned} PFHIWA(\underline{\omega}_1, \underline{\omega}_2 \ldots, \underline{\omega}_{\dot{n}}) &= đ_1\underline{\omega}_1 \oplus đ_2\underline{\omega}_2 \oplus \ldots \oplus đ_{\dot{n}}\underline{\omega}_{\dot{n}} \\ &= đ_1\underline{\omega}_0 \oplus đ_2\underline{\omega}_0 \oplus \ldots \oplus đ_{\dot{n}}\underline{\omega}_0 \\ &= (đ_1 + đ_2 + \cdots + đ_{\dot{n}})\underline{\omega}_0 \\ &= \underline{\omega}_0. \end{aligned}$$

Theorem 3 (Boundedness). Suppose m is the PFHIWA operator, $\underline{\omega}^- = min\{\underline{\omega}_1, \underline{\omega}_2 \ldots, \underline{\omega}_{\dot{n}}\}$, $\underline{\omega}^+ = max\{\underline{\omega}_1, \underline{\omega}_2 \ldots, \underline{\omega}_{\dot{n}}\}$ and $\underline{\omega}^- \leq m(\underline{\omega}_1, \underline{\omega}_2 \ldots, \underline{\omega}_{\dot{n}}) \leq \underline{\omega}^+$.

Proof: Since $min_i\{\underline{\omega}_i\} = \underline{\omega}^-$ and $max_i\{\underline{\omega}_i\} = \underline{\omega}^+$ and m be a PFHIWA operator. So, by Theorem 1, we have

$$m(\underline{\omega}_1, \underline{\omega}_2 \ldots, \underline{\omega}_{\dot{n}}) = \oplus_{i=1}^{\dot{n}} đ_i\underline{\omega}_i \leq \oplus_{i=1}^{\dot{n}} đ_i\underline{\omega}^+ = \underline{\omega}^+ \oplus_{i=1}^{\dot{n}} đ_i$$
$$m(\underline{\omega}_1, \underline{\omega}_2 \ldots, \underline{\omega}_{\dot{n}}) = \oplus_{i=1}^{\dot{n}} đ_i\underline{\omega}_i \geq \oplus_{i=1}^{\dot{n}} đ_i\underline{\omega}^- = \underline{\omega}^- \oplus_{i=1}^{\dot{n}} đ_i$$

Since $\sum_{i=1}^{\dot{n}} đ_i = 1$. So, we have

$m(\underline{\omega}_1, \underline{\omega}_2 \ldots, \underline{\omega}_{\dot{n}}) \leq \underline{\omega}^+$ and $m(\underline{\omega}_1, \underline{\omega}_2 \ldots, \underline{\omega}_{\dot{n}}) \geq \underline{\omega}^-$

Therefore, $\underline{\omega}^- \leq m(\underline{\omega}_1, \underline{\omega}_2 \ldots, \underline{\omega}_{\dot{n}}) \leq \underline{\omega}^+$.

Theorem 4 (Monotonicity). For collections of two different PyFNs u_i and v_i such that $u_i \leq v_i$ for all i and m to be a PFHIWA operator, then $m(u_1, u_2 \ldots, u_{\dot{n}}) \leq m(v_1, v_2 \ldots, v_{\dot{n}})$.

Proof: The proof is the same as above.

We will show the PFH-ordered weighted averaging operator (PFHIOWA).

Definition 12. Suppose Y be the collection of PyNFs and $PFHIOWA: \Upsilon^{\dot{n}} \to \Upsilon$, such that:

$$PFHIOWA(\underline{\omega}_1, \underline{\omega}_2 \ldots, \underline{\omega}_{\dot{n}}) = đ_1\underline{\omega}_{\partial(1)} \oplus đ_2\underline{\omega}_{\partial(2)} \oplus \ldots \oplus đ_{\dot{n}}\underline{\omega}_{\partial(\dot{n})}$$

Where ∂ be the permutation such that $\underline{\omega}_{\partial(i-1)} \leq \underline{\omega}_{\partial(i)}(i = 1,2, \ldots, \dot{n})$ and $đ_i$ is the normalized weight vector of $\underline{\omega}_i$.

Theorem 5. For PyFNs $\underline{\omega}_i = \langle g_i, h_i \rangle$ with $g_i \neq 1(i = 1,2, \ldots, \dot{n})$, the aggregated value by using PFHIOWA is again a PyFN and is defined as:

$$PFHIOWA(\underline{\omega}_1, \underline{\omega}_2 \ldots, \underline{\omega}_{\dot{n}}) =$$

$$\left(\sqrt{\frac{\oplus_{\mathrm{i}=1}^{\dot{n}}\left[1+(\mathfrak{Y}-1)(g_{\partial(\mathrm{i})})^2\right]^{đ_\mathrm{i}}-\oplus_{\mathrm{i}=1}^{\dot{n}}\left(1-(g_{\partial(\mathrm{i})})^2\right)^{đ_\mathrm{i}}}{\oplus_{\mathrm{i}=1}^{\dot{n}}\left[1+(\mathfrak{Y}-1)(g_{\partial(\mathrm{i})})^2\right]^{đ_\mathrm{i}}-(\mathfrak{Y}-1)\oplus_{\mathrm{i}=1}^{\dot{n}}\left(1-(g_{\partial(\mathrm{i})})^2\right)^{đ_\mathrm{i}}}}, \sqrt{\frac{\mathfrak{Y}\oplus_{\mathrm{i}=1}^{\dot{n}}\left(1-(g_{\partial(\mathrm{i})})^2\right)^{đ_\mathrm{i}}-\mathfrak{Y}\oplus_{\mathrm{i}=1}^{\dot{n}}\left(1-(g_{\partial(\mathrm{i})})^2-(\mathrm{h}_{\partial(\mathrm{i})})^2\right)^{đ_\mathrm{i}}}{\oplus_{\mathrm{i}=1}^{\dot{n}}\left[1+(\mathfrak{Y}-1)(g_{\partial(\mathrm{i})})^2\right]^{đ_\mathrm{i}}-(\mathfrak{Y}-1)\oplus_{\mathrm{i}=1}^{\dot{n}}\left(1-(g_{\partial(\mathrm{i})})^2\right)^{đ_\mathrm{i}}}}\right).$$

Properties for PFHIOWA are defined below.

Theorem 6 (Idempotency). If $\underline{\omega}_\mathrm{i}=\underline{\omega}$ for all i then $PFHIOWA(\underline{\omega}_1,\underline{\omega}_2\ldots,\underline{\omega}_{\dot{n}})=\underline{\omega}$.

Theorem 7 (*Boundedness*). Suppose m is the PFHIOWA operator, $\underline{\omega}^-=min\{\underline{\omega}_1,\underline{\omega}_2\ldots,\underline{\omega}_{\dot{n}}\}$, $\underline{\omega}^+=max\{\underline{\omega}_1,\underline{\omega}_2\ldots,\underline{\omega}_{\dot{n}}\}$ and $\underline{\omega}^-\le m(\underline{\omega}_1,\underline{\omega}_2\ldots,\underline{\omega}_{\dot{n}})\le\underline{\omega}^+$.

Theorem 8 (*Monotonicity*). For collections of two different PyFNs u_i and v_i such that $u_\mathrm{i}\le v_\mathrm{i}$ for all i and m to be a PFHIOWA operator, then $m(u_1,u_2\ldots,u_{\dot{n}})\le m(v_1,v_2\ldots,v_{\dot{n}})$.v

3.2 Geometric Operators

For the set of $\underline{\omega}_\mathrm{i}=\langle g_\mathrm{i},\mathrm{h}_\mathrm{i}\rangle(\mathrm{i}=1,2,\ldots,\dot{n})$ with $\mathrm{h}_\mathrm{i}\ne 1$ for all i and relying on the above rules, we will present new PFH interactive weighted (PFHIWG) and order weighted (PFHIOWG) geometric operators.

Definition 13. Suppose Y be the collection of PyNFs and $PFHIWA:\Upsilon^{\dot{n}}\to\Upsilon$, such that:

$$PFHIWA(\underline{\omega}_1,\underline{\omega}_2\ldots,\underline{\omega}_{\dot{n}})=đ_1\underline{\omega}_1\otimes đ_2\underline{\omega}_2\otimes\ldots\otimes đ_{\dot{n}}\underline{\omega}_{\dot{n}}$$

where $đ=(đ_1,đ_2,\ldots,đ_{\dot{n}})^T$ is the weight vector of $\underline{\omega}_\mathrm{i}$ such that $đ_\mathrm{i}>0$ and $\sum_{\mathrm{i}=1}^{\dot{n}}đ_\mathrm{i}=1$; then, PFHIWG is defined as PFH interactive weighted geometric operator.

Theorem 9. For PyFNs $\underline{\omega}_\mathrm{i}=\langle g_\mathrm{i},\mathrm{h}_\mathrm{i}\rangle$ with $g_\mathrm{i}\ne 1(\mathrm{i}=1,2,\ldots,\dot{n})$, the aggregated value by using PFHIWG is again a PyFN and is defined as:

$$PFHIWG(\underline{\omega}_1,\underline{\omega}_2\ldots,\underline{\omega}_{\dot{n}})=$$

$$\left(\sqrt{\frac{\mathfrak{Y}\oplus_{\mathrm{i}=1}^{\dot{n}}(1-(\mathrm{h}_\mathrm{i})^2)^{đ_\mathrm{i}}-\mathfrak{Y}\oplus_{\mathrm{i}=1}^{\dot{n}}(1-(g_\mathrm{i})^2-(\mathrm{h}_\mathrm{i})^2)^{đ_\mathrm{i}}}{\oplus_{\mathrm{i}=1}^{\dot{n}}[1+(\mathfrak{Y}-1)(\mathrm{h}_\mathrm{i})^2]^{đ_\mathrm{i}}-(\mathfrak{Y}-1)\oplus_{\mathrm{i}=1}^{\dot{n}}(1-(\mathrm{h}_\mathrm{i})^2)^{đ_\mathrm{i}}}}, \sqrt{\frac{\oplus_{\mathrm{i}=1}^{\dot{n}}[1+(\mathfrak{Y}-1)(\mathrm{h}_\mathrm{i})^2]^{đ_\mathrm{i}}-\oplus_{\mathrm{i}=1}^{\dot{n}}(1-(\mathrm{h}_\mathrm{i})^2)^{đ_\mathrm{i}}}{\oplus_{\mathrm{i}=1}^{\dot{n}}[1+(\mathfrak{Y}-1)(\mathrm{h}_\mathrm{i})^2]^{đ_\mathrm{i}}-(\mathfrak{Y}-1)\oplus_{\mathrm{i}=1}^{\dot{n}}(1-(\mathrm{h}_\mathrm{i})^2)^{đ_\mathrm{i}}}}\right) \quad (5)$$

Proof: We will prove *Theorem 9* by mathematical induction. So, we have

I. When $\dot{n}=2$, we have $g_1+g_2=1$ and $PFHIWG(\underline{\omega}_1,\underline{\omega}_2)=đ_1\underline{\omega}_1\oplus đ_2\underline{\omega}_2$. Now, for PyFNs $\underline{\omega}_1$ and $\underline{\omega}_2$, we have

$$đ_1\underline{\omega}_1=\left(\sqrt{\frac{\mathfrak{Y}\left(1-(\mathrm{h}_1)^2\right)^{đ_1}-\mathfrak{Y}\left(1-(g_1)^2-(\mathrm{h}_1)^2\right)^{đ_1}}{\left[1+(\mathfrak{Y}-1)(\mathrm{h}_1)^2\right]^{đ_1}-(\mathfrak{Y}-1)\left(1-(\mathrm{h}_1)^2\right)^{đ_1}}}, \sqrt{\frac{\left[1+(\mathfrak{Y}-1)(\mathrm{h}_1)^2\right]^{đ_1}-\left(1-(\mathrm{h}_1)^2\right)^{đ_1}}{\left[1+(\mathfrak{Y}-1)(\mathrm{h}_1)^2\right]^{đ_1}-(\mathfrak{Y}-1)\left(1-(\mathrm{h}_1)^2\right)^{đ_1}}}\right)$$

and

$$đ_2\underline{\omega}_2=\left(\sqrt{\frac{\mathfrak{Y}\left(1-(\mathrm{h}_2)^2\right)^{đ_1}-\mathfrak{Y}\left(1-(g_2)^2-(\mathrm{h}_2)^2\right)^{đ_1}}{\left[1+(\mathfrak{Y}-1)(\mathrm{h}_2)^2\right]^{đ_1}-(\mathfrak{Y}-1)\left(1-(\mathrm{h}_2)^2\right)^{đ_1}}}, \sqrt{\frac{\left[1+(\mathfrak{Y}-1)(\mathrm{h}_2)^2\right]^{đ_1}-\left(1-(\mathrm{h}_2)^2\right)^{đ_1}}{\left[1+(\mathfrak{Y}-1)(\mathrm{h}_2)^2\right]^{đ_1}-(\mathfrak{Y}-1)\left(1-(\mathrm{h}_2)^2\right)^{đ_1}}}\right)$$

So, we have

$$PFHIWG(\underline{\omega}_1,\underline{\omega}_2)=đ_1\underline{\omega}_1\otimes đ_2\underline{\omega}_2$$

$$= \left(\sqrt{\frac{\mathfrak{Y}(1-(\mathrm{h}_1)^2)^{\text{đ}_1} - \mathfrak{Y}(1-(g_1)^2-(\mathrm{h}_1)^2)^{\text{đ}_1}}{[1+(\mathfrak{Y}-1)(\mathrm{h}_1)^2]^{\text{đ}_1} - (\mathfrak{Y}-1)(1-(\mathrm{h}_1)^2)^{\text{đ}_1}}}, \sqrt{\frac{[1+(\mathfrak{Y}-1)(\mathrm{h}_1)^2]^{\text{đ}_1} - (1-(\mathrm{h}_1)^2)^{\text{đ}_1}}{[1+(\mathfrak{Y}-1)(\mathrm{h}_1)^2]^{\text{đ}_1} - (\mathfrak{Y}-1)(1-(\mathrm{h}_1)^2)^{\text{đ}_1}}} \right)$$

$$\oplus \left(\sqrt{\frac{\mathfrak{Y}(1-(\mathrm{h}_2)^2)^{\text{đ}_1} - \mathfrak{Y}(1-(g_2)^2-(\mathrm{h}_2)^2)^{\text{đ}_1}}{[1+(\mathfrak{Y}-1)(\mathrm{h}_2)^2]^{\text{đ}_1} - (\mathfrak{Y}-1)(1-(\mathrm{h}_2)^2)^{\text{đ}_1}}}, \sqrt{\frac{[1+(\mathfrak{Y}-1)(\mathrm{h}_2)^2]^{\text{đ}_1} - (1-(\mathrm{h}_2)^2)^{\text{đ}_1}}{[1+(\mathfrak{Y}-1)(\mathrm{h}_2)^2]^{\text{đ}_1} - (\mathfrak{Y}-1)(1-(\mathrm{h}_2)^2)^{\text{đ}_1}}} \right)$$

$$= \left(\sqrt{\frac{\mathfrak{Y} \oplus_{i=1}^{2} (1-(\mathrm{h}_i)^2)^{\text{đ}_i} - \mathfrak{Y} \oplus_{i=1}^{2} (1-(g_i)^2-(\mathrm{h}_i)^2)^{\text{đ}_i}}{\oplus_{i=1}^{2} [1+(\mathfrak{Y}-1)(\mathrm{h}_i)^2]^{\text{đ}_i} - (\mathfrak{Y}-1) \oplus_{i=1}^{2} (1-(\mathrm{h}_i)^2)^{\text{đ}_i}}}, \sqrt{\frac{\oplus_{i=1}^{2} [1+(\mathfrak{Y}-1)(\mathrm{h}_i)^2]^{\text{đ}_i} - \oplus_{i=1}^{2} (1-(\mathrm{h}_i)^2)^{\text{đ}_i}}{\oplus_{i=1}^{2} [1+(\mathfrak{Y}-1)(\mathrm{h}_i)^2]^{\text{đ}_i} - (\mathfrak{Y}-1) \oplus_{i=1}^{2} (1-(\mathrm{h}_i)^2)^{\text{đ}_i}}} \right)$$

So, Eq. (5) is true for $\dot{n} = 2$

II. Suppose that Eq. (5) is also true for $\dot{n} = \kappa$

$$PFHIWG(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}})$$

$$= \left(\sqrt{\frac{\mathfrak{Y} \oplus_{i=1}^{\kappa} (1-(\mathrm{h}_i)^2)^{\text{đ}_i} - \mathfrak{Y} \oplus_{i=1}^{\kappa} (1-(g_i)^2-(\mathrm{h}_i)^2)^{\text{đ}_i}}{\oplus_{i=1}^{\kappa} [1+(\mathfrak{Y}-1)(\mathrm{h}_i)^2]^{\text{đ}_i} - (\mathfrak{Y}-1) \oplus_{i=1}^{\kappa} (1-(\mathrm{h}_i)^2)^{\text{đ}_i}}}, \sqrt{\frac{\oplus_{i=1}^{\kappa} [1+(\mathfrak{Y}-1)(\mathrm{h}_i)^2]^{\text{đ}_i} - \oplus_{i=1}^{\kappa} (1-(\mathrm{h}_i)^2)^{\text{đ}_i}}{\oplus_{i=1}^{\kappa} [1+(\mathfrak{Y}-1)(\mathrm{h}_i)^2]^{\text{đ}_i} - (\mathfrak{Y}-1) \oplus_{i=1}^{\kappa} (1-(\mathrm{h}_i)^2)^{\text{đ}_i}}} \right)$$

III. Now, for $\dot{n} = \kappa + 1$, we have

$$PFHIWG(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\kappa+1}) = \oplus_{i=1}^{\kappa+1} \text{đ}_i \underline{\omega}_i = PFHIWG(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\kappa}) \oplus \text{đ}_{\kappa+1} \underline{\omega}_{\kappa+1}$$

$$= \left(\sqrt{\frac{\mathfrak{Y} \oplus_{i=1}^{\kappa} (1-(\mathrm{h}_i)^2)^{\text{đ}_i} - \mathfrak{Y} \oplus_{i=1}^{\kappa} (1-(g_i)^2-(\mathrm{h}_i)^2)^{\text{đ}_i}}{\oplus_{i=1}^{\kappa} [1+(\mathfrak{Y}-1)(\mathrm{h}_i)^2]^{\text{đ}_i} - (\mathfrak{Y}-1) \oplus_{i=1}^{\kappa} (1-(\mathrm{h}_i)^2)^{\text{đ}_i}}}, \sqrt{\frac{\oplus_{i=1}^{\kappa} [1+(\mathfrak{Y}-1)(\mathrm{h}_i)^2]^{\text{đ}_i} - \oplus_{i=1}^{\kappa} (1-(\mathrm{h}_i)^2)^{\text{đ}_i}}{\oplus_{i=1}^{\kappa} [1+(\mathfrak{Y}-1)(\mathrm{h}_i)^2]^{\text{đ}_i} - (\mathfrak{Y}-1) \oplus_{i=1}^{\kappa} (1-(\mathrm{h}_i)^2)^{\text{đ}_i}}} \right)$$

$$\oplus \left(\sqrt{\frac{\mathfrak{Y}(1-(\mathrm{h}_{\kappa+1})^2)^{\text{đ}_{\kappa+1}} - \mathfrak{Y}(1-(g_{\kappa+1})^2-(\mathrm{h}_{\kappa+1})^2)^{\text{đ}_{\kappa+1}}}{[1+(\mathfrak{Y}-1)(\mathrm{h}_{\kappa+1})^2]^{\text{đ}_{\kappa+1}} - (\mathfrak{Y}-1)(1-(\mathrm{h}_{\kappa+1})^2)^{\text{đ}_{\kappa+1}}}}, \sqrt{\frac{[1+(\mathfrak{Y}-1)(\mathrm{h}_{\kappa+1})^2]^{\text{đ}_{\kappa+1}} - (1-(\mathrm{h}_{\kappa+1})^2)^{\text{đ}_{\kappa+1}}}{[1+(\mathfrak{Y}-1)(\mathrm{h}_{\kappa+1})^2]^{\text{đ}_{\kappa+1}} - (\mathfrak{Y}-1)(1-(\mathrm{h}_{\kappa+1})^2)^{\text{đ}_{\kappa+1}}}} \right)$$

$$= \left(\sqrt{\frac{\mathfrak{Y} \oplus_{i=1}^{\kappa+1} (1-(\mathrm{h}_i)^2)^{\text{đ}_i} - \mathfrak{Y} \oplus_{i=1}^{\kappa+1} (1-(g_i)^2-(\mathrm{h}_i)^2)^{\text{đ}_i}}{\oplus_{i=1}^{\kappa+1} [1+(\mathfrak{Y}-1)(\mathrm{h}_i)^2]^{\text{đ}_i} - (\mathfrak{Y}-1) \oplus_{i=1}^{\kappa+1} (1-(\mathrm{h}_i)^2)^{\text{đ}_i}}}, \sqrt{\frac{\oplus_{i=1}^{\kappa+1} [1+(\mathfrak{Y}-1)(\mathrm{h}_i)^2]^{\text{đ}_i} - \oplus_{i=1}^{\kappa+1} (1-(\mathrm{h}_i)^2)^{\text{đ}_i}}{\oplus_{i=1}^{\kappa+1} [1+(\mathfrak{Y}-1)(\mathrm{h}_i)^2]^{\text{đ}_i} - (\mathfrak{Y}-1) \oplus_{i=1}^{\kappa+1} (1-(\mathrm{h}_i)^2)^{\text{đ}_i}}} \right)$$

So, Eq. (5) also holds for $\dot{n} = \kappa + 1$.

Hence, it is proved that the Eq. (5) holds for all positive integer $\dot{n}$.

Now, we will define some properties for PFHIWG.

Theorem 10 (*Idempotency*). If $\underline{\omega}_i = \underline{\omega}_0$ for all i then $PFHIWG(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) = \underline{\omega}_0$.

Proof: For a set of PyFNs $\underline{\omega}_i (i = 1,2, \dots, \dot{n})$ and weight vector đ_i such that $\text{đ}_i > 0$ and $\sum_{i=1}^{\dot{n}} \text{đ}_i = 1$. Suppose $\underline{\omega}_i = \underline{\omega}_0$ for all i then by PFHIWG operator, we have

$$\begin{aligned} PFHIWA(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) &= \text{đ}_1 \underline{\omega}_1 \oplus \text{đ}_2 \underline{\omega}_2 \oplus \dots \oplus \text{đ}_{\dot{n}} \underline{\omega}_{\dot{n}} \\ &= \text{đ}_1 \underline{\omega}_0 \oplus \text{đ}_2 \underline{\omega}_0 \oplus \dots \oplus \text{đ}_{\dot{n}} \underline{\omega}_0 \\ &= (\text{đ}_1 + \text{đ}_2 + \dots + \text{đ}_{\dot{n}}) \underline{\omega}_0 \\ &= \underline{\omega}_0. \end{aligned}$$

Theorem 11 (Boundedness). Suppose m is the PFHIWG operator, $\underline{\omega}^- = min\{\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}\}$, $\underline{\omega}^+ = max\{\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}\}$ and $\underline{\omega}^- \le m(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) \le \underline{\omega}^+$.

Proof: Since $min_i\{\underline{\omega}_i\} = \underline{\omega}^-$ and $max_i\{\underline{\omega}_i\} = \underline{\omega}^+$ and m be a PFHIWG operator. So, by Theorem 1, we have

$$m(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) = \oplus_{i=1}^{\dot{n}} \text{đ}_i \underline{\omega}_i \le \oplus_{i=1}^{\dot{n}} \text{đ}_i \underline{\omega}^+ = \underline{\omega}^+ \oplus_{i=1}^{\dot{n}} \text{đ}_i$$

$$m(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) = \oplus_{i=1}^{\dot{n}} \text{đ}_i \underline{\omega}_i \ge \oplus_{i=1}^{\dot{n}} \text{đ}_i \underline{\omega}^- = \underline{\omega}^- \oplus_{i=1}^{\dot{n}} \text{đ}_i$$

Since $\sum_{i=1}^{\dot{n}} \text{đ}_i = 1$. So, we have

$m(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) \le \underline{\omega}^+$ and $m(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) \ge \underline{\omega}^-$

Therefore, $\underline{\omega}^- \le m(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) \le \underline{\omega}^+$.

Theorem 12 (Monotonicity). For collections of two different PyFNs u_i and v_i such that $u_i \le v_i$ for all i and m to be a PFHIWG operator, then $m(u_1, u_2 \dots, u_{\dot{n}}) \le m(v_1, v_2 \dots, v_{\dot{n}})$.

Proof: The proof is the same as above.

Now, we will show the PFH-ordered weighted geometric operator (PFHIOWG).

Definition 14. Suppose Y be the collection of PyNFs and $PFHIOWG: \Upsilon^{\dot{n}} \rightarrow \Upsilon$, such that:

$$PFHIOWG(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) = đ_1\underline{\omega}_{\partial(1)} \oplus đ_2\underline{\omega}_{\partial(2)} \oplus \dots \oplus đ_{\dot{n}}\underline{\omega}_{\partial(\dot{n})}$$

Where ∂ be the permutation such that $\underline{\omega}_{\partial(\mathfrak{i}-1)} \leq \underline{\omega}_{\partial(\mathfrak{i})}(\mathfrak{i} = 1,2, \dots, \dot{n})$ and $đ_{\mathfrak{i}}$ is the normalized weight vector of $\underline{\omega}_{\mathfrak{i}}$.

Theorem 13. For PyFNs $\underline{\omega}_{\mathfrak{i}} = \langle g_{\mathfrak{i}}, h_{\mathfrak{i}} \rangle$ with $g_{\mathfrak{i}} \neq 1(\mathfrak{i} = 1,2, \dots, \dot{n})$, the aggregated value by using PFHIOWG is again a PyFN and is defined as:

$$PFHIOWG(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) =$$

$$\left(\sqrt{\frac{\mathfrak{Y} \oplus_{\mathfrak{i}=1}^{\dot{n}} \left(1 - \left(h_{\partial(\mathfrak{i})}\right)^2\right)^{đ_{\mathfrak{i}}} - \mathfrak{Y} \oplus_{\mathfrak{i}=1}^{\dot{n}} \left(1 - \left(g_{\partial(\mathfrak{i})}\right)^2 - \left(h_{\partial(\mathfrak{i})}\right)^2\right)^{đ_{\mathfrak{i}}}}{\oplus_{\mathfrak{i}=1}^{\dot{n}} \left[1 + (\mathfrak{Y} - 1)h_{\partial(\mathfrak{i})}\right]^{đ_{\mathfrak{i}}} - (\mathfrak{Y} - 1) \oplus_{\mathfrak{i}=1}^{\dot{n}} \left(1 - \left(h_{\partial(\mathfrak{i})}\right)^2\right)^{đ_{\mathfrak{i}}}}}, \sqrt{\frac{\oplus_{\mathfrak{i}=1}^{\dot{n}} \left[1 + (\mathfrak{Y} - 1)\left(h_{\partial(\mathfrak{i})}\right)^2\right]^{đ_{\mathfrak{i}}} - \oplus_{\mathfrak{i}=1}^{\dot{n}} \left(1 - \left(h_{\partial(\mathfrak{i})}\right)^2\right)^{đ_{\mathfrak{i}}}}{\oplus_{\mathfrak{i}=1}^{\dot{n}} \left[1 + (\mathfrak{Y} - 1)\left(h_{\partial(\mathfrak{i})}\right)^2\right]^{đ_{\mathfrak{i}}} - (\mathfrak{Y} - 1) \oplus_{\mathfrak{i}=1}^{\dot{n}} \left(1 - \left(h_{\partial(\mathfrak{i})}\right)^2\right)^{đ_{\mathfrak{i}}}}} \right).$$

Properties for PFHIOWG are defined below.

Theorem 14 (Idempotency). If $\underline{\omega}_{\mathfrak{i}} = \underline{\omega}$ for all $\mathfrak{i}$ then $PFHIOWG(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) = \underline{\omega}$.

Theorem 15 (Boundedness). Suppose m is the PFHIOWG operator, $\underline{\omega}^- = min\{\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}\}$, $\underline{\omega}^+ = max\{\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}\}$ and $\underline{\omega}^- \leq m(\underline{\omega}_1, \underline{\omega}_2 \dots, \underline{\omega}_{\dot{n}}) \leq \underline{\omega}^+$.

Theorem 16 (Monotonicity). For collections of two different PyFNs $u_{\mathfrak{i}}$ and $v_{\mathfrak{i}}$ such that $u_{\mathfrak{i}} \leq v_{\mathfrak{i}}$ for all $\mathfrak{i}$ and m to be a PFHIOWG operator, then $m(u_1, u_2 \dots, u_{\dot{n}}) \leq m(v_1, v_2 \dots, v_{\dot{n}})$.

4. Multi-attribute Decision Making Technique

This section will show a DM method for solving the MADM issue under PyF circumstances. In everyday life, DM involves selecting the best option from a set of choices. Typically, decision-makers rely on intuition and expertise. However, because decision-making problems can be complex, accurately representing fuzzy and vague information is crucial. To capture decision makers' preferences, preference relations are valuable tools for ranking criteria. For a set of criteria V={V_1, $V_{2,\dots,}$ V_n}, experts compare each pair and construct preference relations.

We will discuss the steps involved in the MADM technique.

Step 1. First, we construct a PyF data decision matrix. This matrix is then converted into the normalized matrix.

Step 2. Then, using our proposed operator, all the PyF data is converted into a single value.

Step 3. Find the score values by using the SF. If the score values are the same, we find accuracy using the AF. So, the ranking will be based on accuracy values.

Step 4. The best choice is found according to the ranking of the options.

Step 5. End.

The flowchart for the MADM is shown in Figure 2.

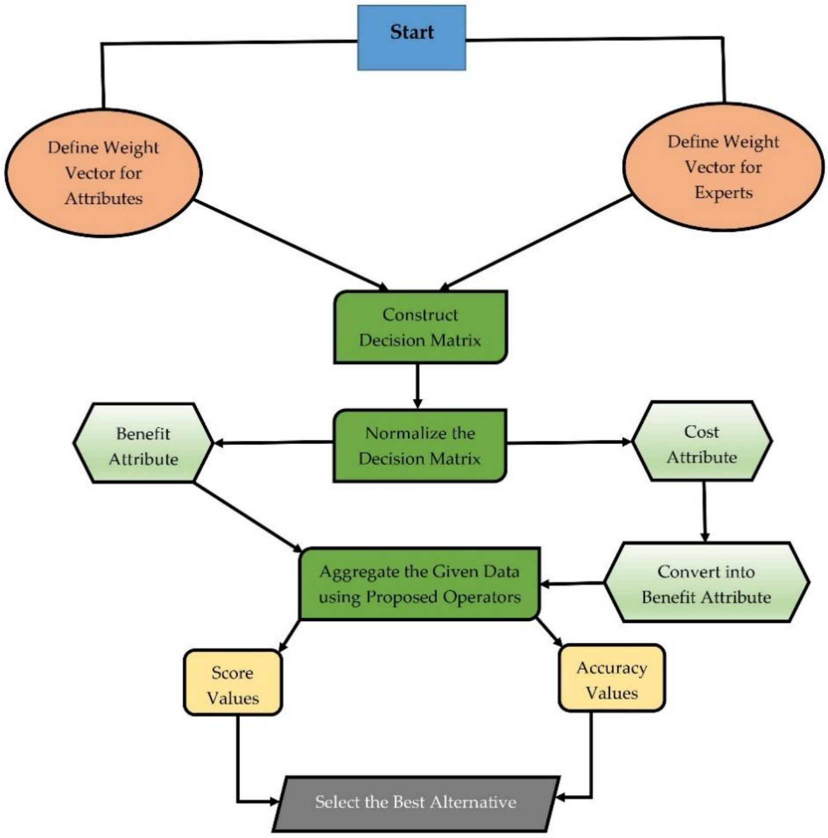

Fig. 2. Flowchart for MADM technique

5. Practical Example

This scenario involves an enterprise located in the industrial zone of Juárez, Mexico, specifically within the Maquiladoras sector, specializing in assembly manufacturing. The company has launched a project aimed at cost reduction, identifying packaging as a significant area for potential savings. To achieve this goal, the company plans to evaluate five suppliers providing packaging for electronic components. Two decision-makers have been invited to participate in the evaluation process. Four criteria (cost, service, lead time, and quality) have been established to evaluate and thoroughly compare the suppliers' offerings to ensure a comprehensive assessment. Therefore, the group of suppliers is denoted as P = {P_1, P_2, P_3, P_4}. The decision maker assigns attribute weights as follows: λ = (0.18, 0.24, 0.33, 0.25). Evaluating the four candidates will involve assessing ambiguity using PyF data across the four attributes outlined in Table 1.

Table 1
PyF matrix for suppliers

P_1	(0.45,0.35)	(0.38,0.53)	(0.83,0.48)	(0.56,0.51)
P_2	(0.67,0.52)	(0.92,0.34)	(0.71,0.61)	(0.37,0.25)
P_3	(0.73,0.35)	(0.52,0.62)	(0.21,0.18)	(0.46,0.24)
P_4	(0.36,0.32)	(0.67,0.45)	(0.52,0.43)	(0.21,0.20)

Now, after using our proposed PFHIWA on the given data, we get the results listed in Table 2.

Table 2
Results by PFHIWA

Operator	PFHIWA
P_1	(0.63804, 0.54359)
P_2	(0.73423, 0.52882)
P_3	(0.48701, 0.42121)
P_4	(0.48724, 0.40315)

By using Eq. (1), we will find the score value, Table 3.

Table 3
Score Values

Operator	PFHIWA
P_1	0.1116
P_2	0.2594
P_3	0.05976
P_4	0.07487

So, by score value, we get the ranking as $P_2 \succ P_1 \succ P_4 \succ P_3$. So, P_2 is selected as the best supplier. Results are also shown graphically in Figure 3.

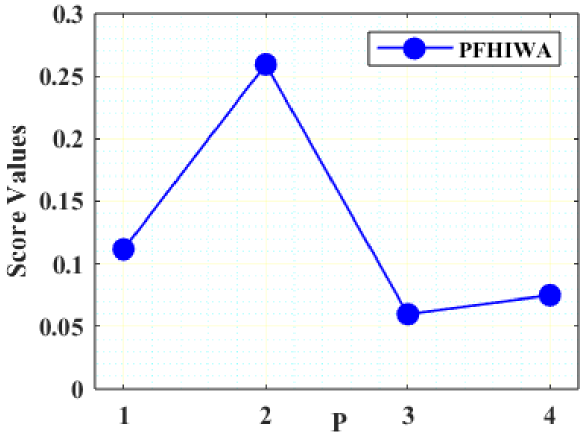

Fig. 3. Score Values

6. Sensitivity Analysis

The influence of the variable $\mathfrak{Y}$ on the DM process has been examined in Table 4 by applying different values of $\mathfrak{Y}$ using the provided method. Based on these diverse preferences, decision-makers can choose the option or options that most closely match their satisfaction criteria. The comprehensive ranking of the given alternatives using our proposed operator is then presented in Table 4 and graphically in Figure 4, showcasing the full range of variations.

Table 4
Different values of $\mathfrak{Y}$

$\mathfrak{Y}$	P_1	P_2	P_3	P_4	Ranking
5	0.07634	0.21902	0.03700	0.05658	$P_2 \succ P_1 \succ P_4 \succ P_3$
10	0.05740	0.19696	0.02045	0.04288	$P_2 \succ P_1 \succ P_4 \succ P_3$
15	0.04949	0.18392	0.01190	0.03574	$P_2 \succ P_1 \succ P_4 \succ P_3$
20	0.03574	0.17516	0.00656	0.03119	$P_2 \succ P_1 \succ P_4 \succ P_3$
25	0.02950	0.15906	0.00286	0.02809	$P_2 \succ P_1 \succ P_4 \succ P_3$

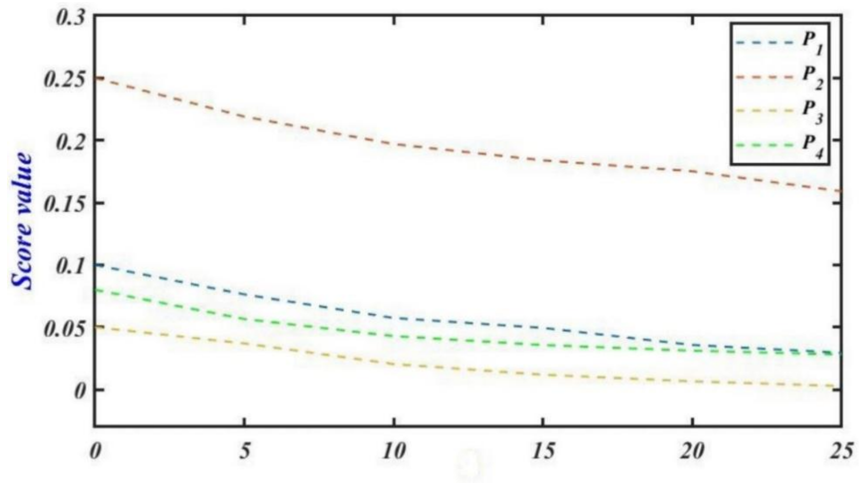

Fig. 4. Different values of $\mathfrak{Y}$.

7. Comparative Analysis

This section will compare our proposed techniques with the existing ones. We have compared our proposed operators with IFHWA [24], IFHWG [29], IFHIWA, and IFHIWG [27]. The results for these operators and our proposed operators are compared in Table 5.

Table 5
Comparative Analysis

Operator	P_1	P_2	P_3	P_4	Ranking
IFHWA	0.3727	0.2767	0.3033	0.2933	$P_1 \succ P_3 \succ P_4 \succ P_2$
IFHWG	0.3393	0.2504	0.3018	0.2476	$P_1 \succ P_3 \succ P_4 \succ P_2$
IFHIWA	0.3536	0.2227	0.2834	0.2639	$P_1 \succ P_3 \succ P_4 \succ P_2$
IFHIWG	0.3530	0.2858	0.3210	0.2806	$P_1 \succ P_3 \succ P_4 \succ P_2$
PFHIWA (Proposed)	0.5946	0.4719	0.5323	0.5137	$P_1 \succ P_3 \succ P_4 \succ P_2$
PFHIWG (Proposed)	0.5941	0.5346	0.5665	0.5297	$P_1 \succ P_3 \succ P_4 \succ P_2$

Figure 5 shows the comparison of our proposed operators with the existing techniques.

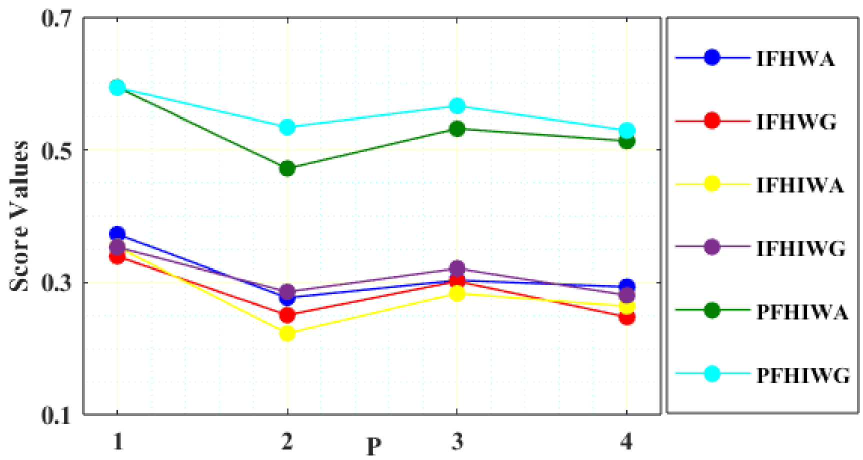

Fig. 5. Comparison Analysis

If we look at Table 5 and Figure 5, then it is clear that our proposed operator is more efficient and shows better results, but the ranking is the same. So, this shows the strength and stability of our proposed operators.

8. Conclusion

This paper aims to introduce Hamacher aggregation operators based on Hamacher t-norm and t-conorm for aggregating PyF information. Firstly, we proposed new operational laws of PFN based on Hamacher t-norm and t-conorm. Then, using these operational laws, we introduce the PFHIWA, PFHIOWA, PFHIWG, and PFHIOWG operators. Additionally, we discussed the fundamental properties of these operators. Subsequently, we presented an algorithm for solving Multiple Attribute Decision Making (MADM) problems under a PyF environment using these operators. Finally, through a numerical example, we demonstrated the sensitivity of the proposed method. The results show that the proposed method offers an improved solution compared to existing methods for ranking options. Future research will extend this work to address multi-objective optimization challenges in various uncertain and fuzzy contexts.

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Conflicts of Interest
The authors declare no conflicts of interest.

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